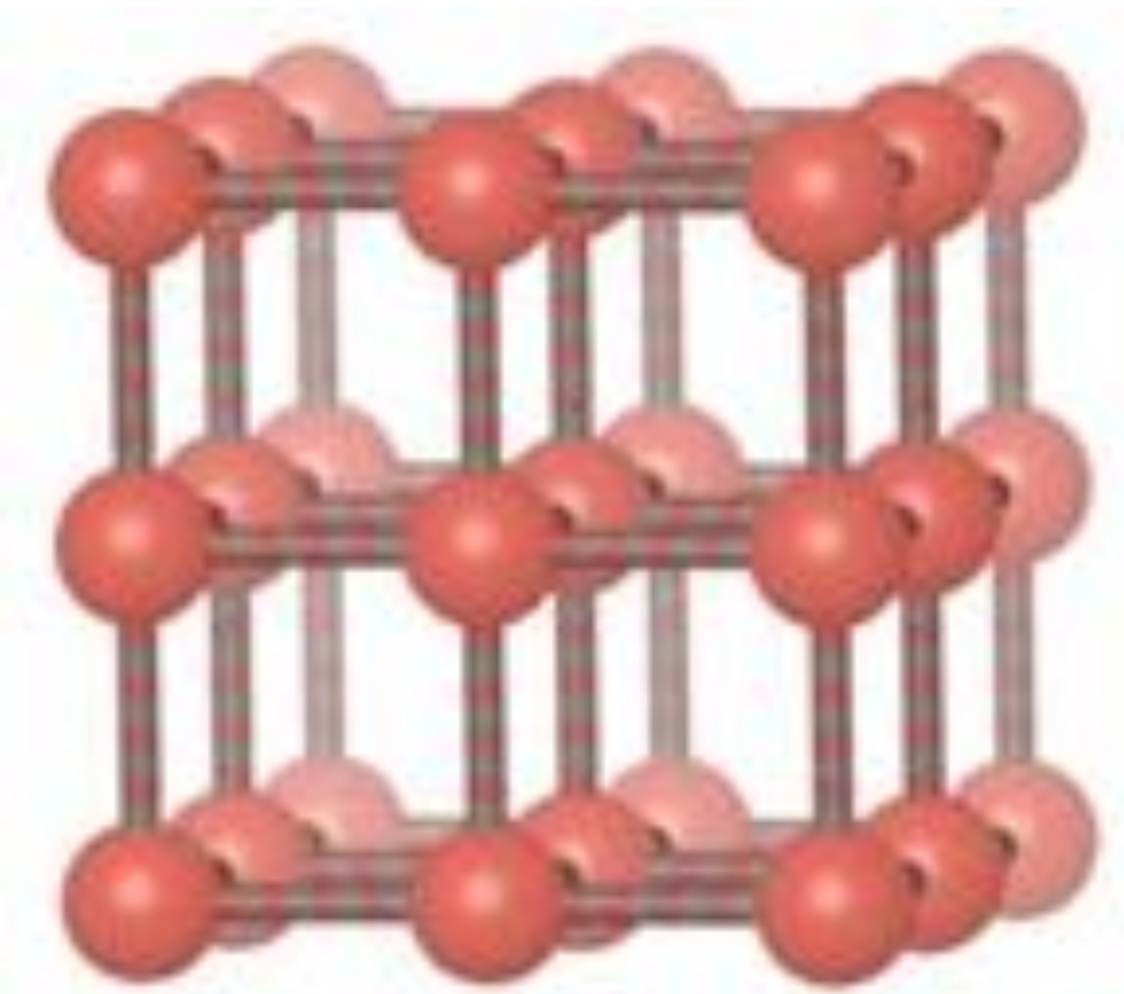
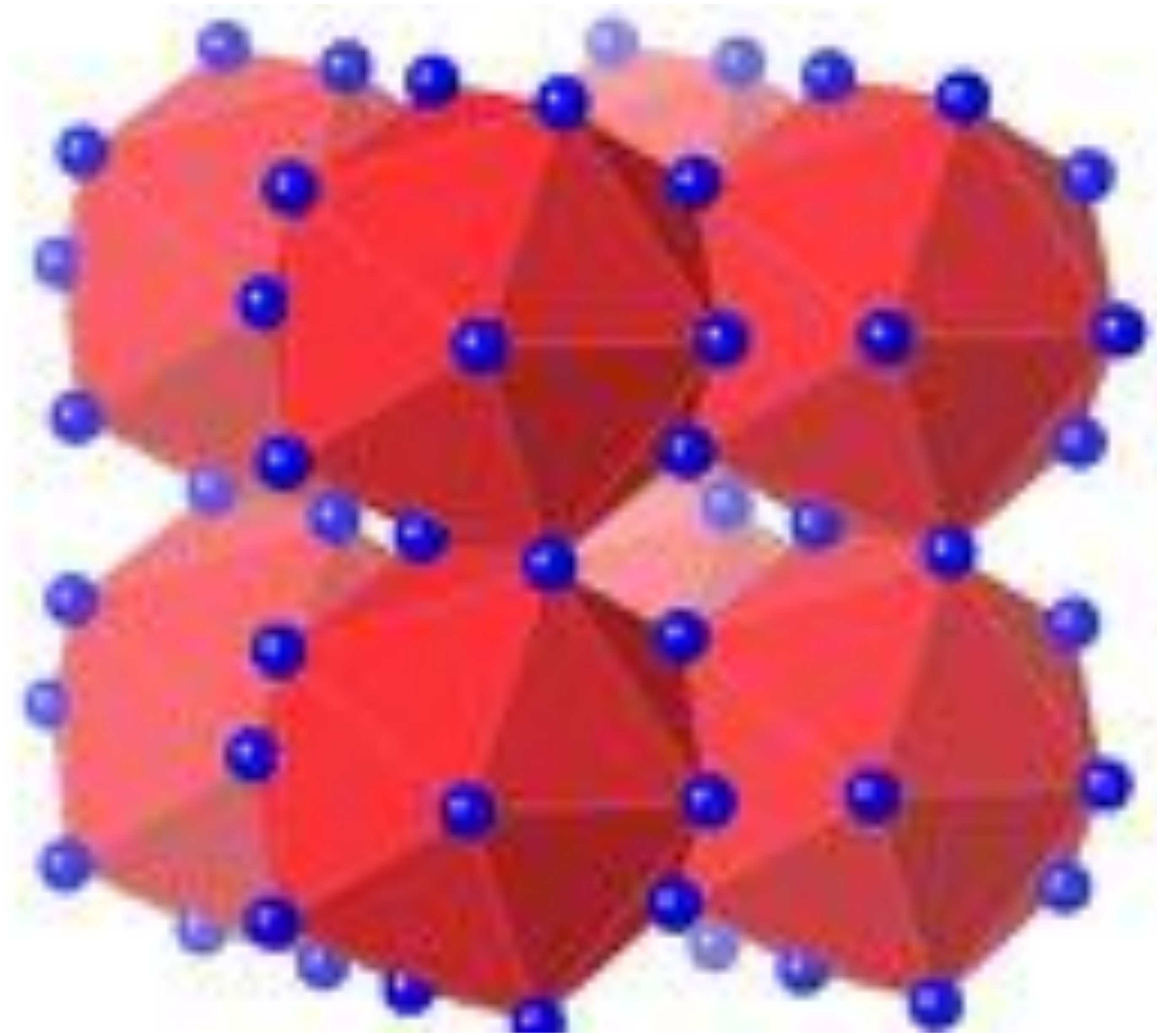
Two-dimensional Crystallography

via topology and orbifolds

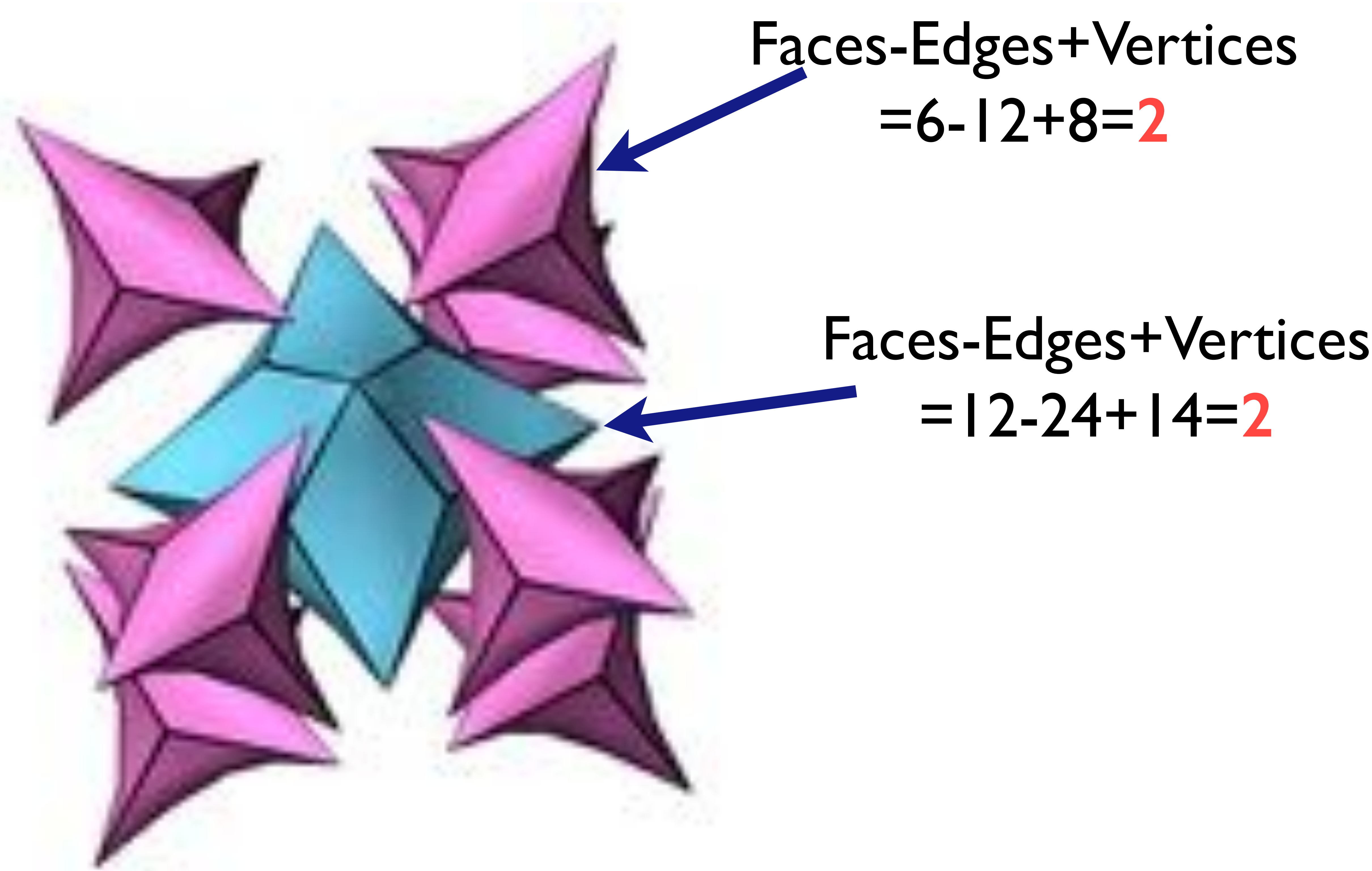
Many 3D structures are also 2D non-euclidean patterns!





- manifolds and Euler characteristic
- construction of manifolds from caps , pants and cross caps
- Gaussian curvature of manifolds
- manifolds as orbifolds: wallpaper

	
faces	4
-edges	-6
vertices	4
Sum	2


Faces-Edges+Vertices
=6-12+8=2
Faces-Edges+Vertices
=12-24+14=2

The face,edge,vertex sum depends on topology only:

$$F - E + V = \chi$$

Euler characteristic

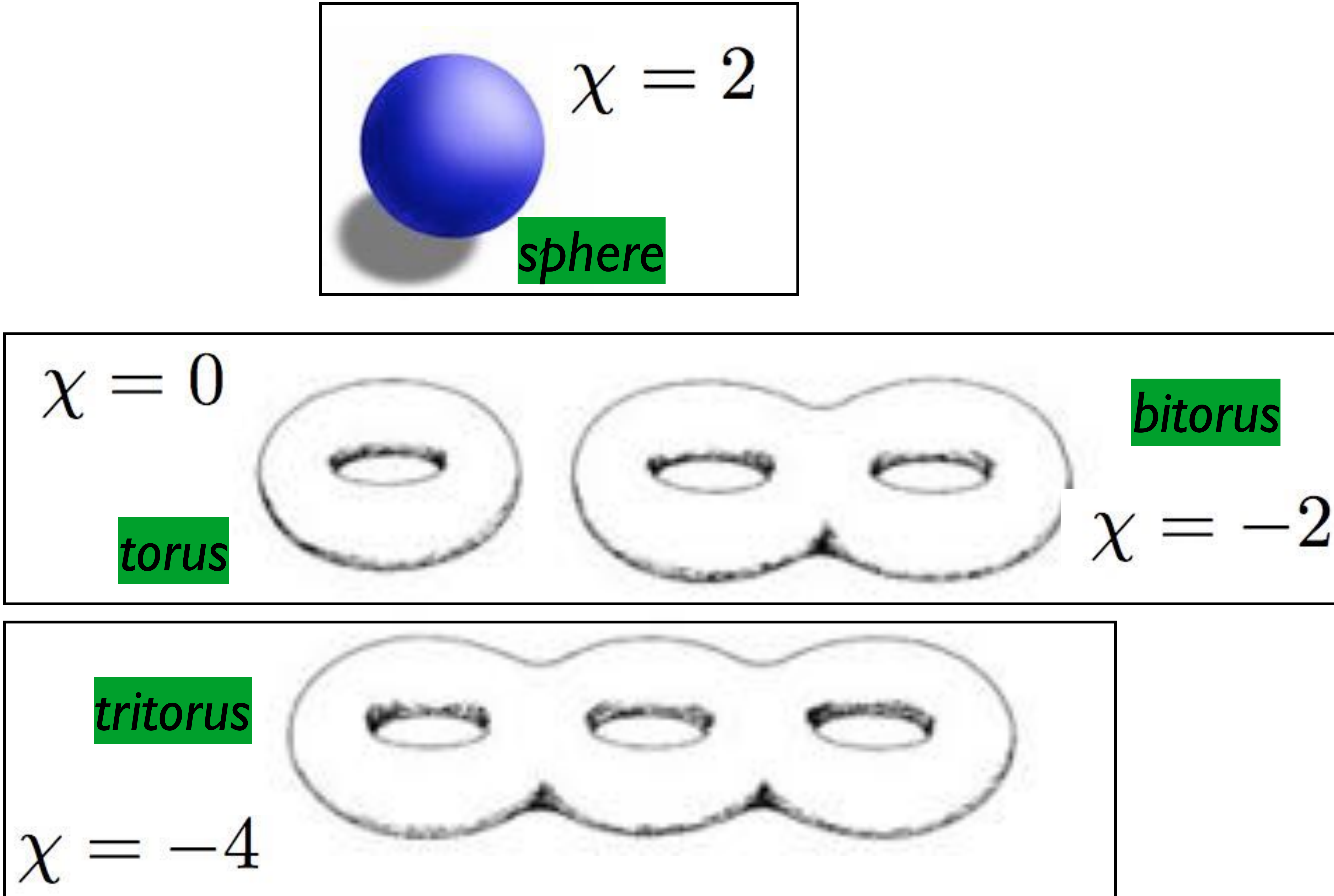
Note: Faces *must be topological* discs

“toroidal polyhedra”

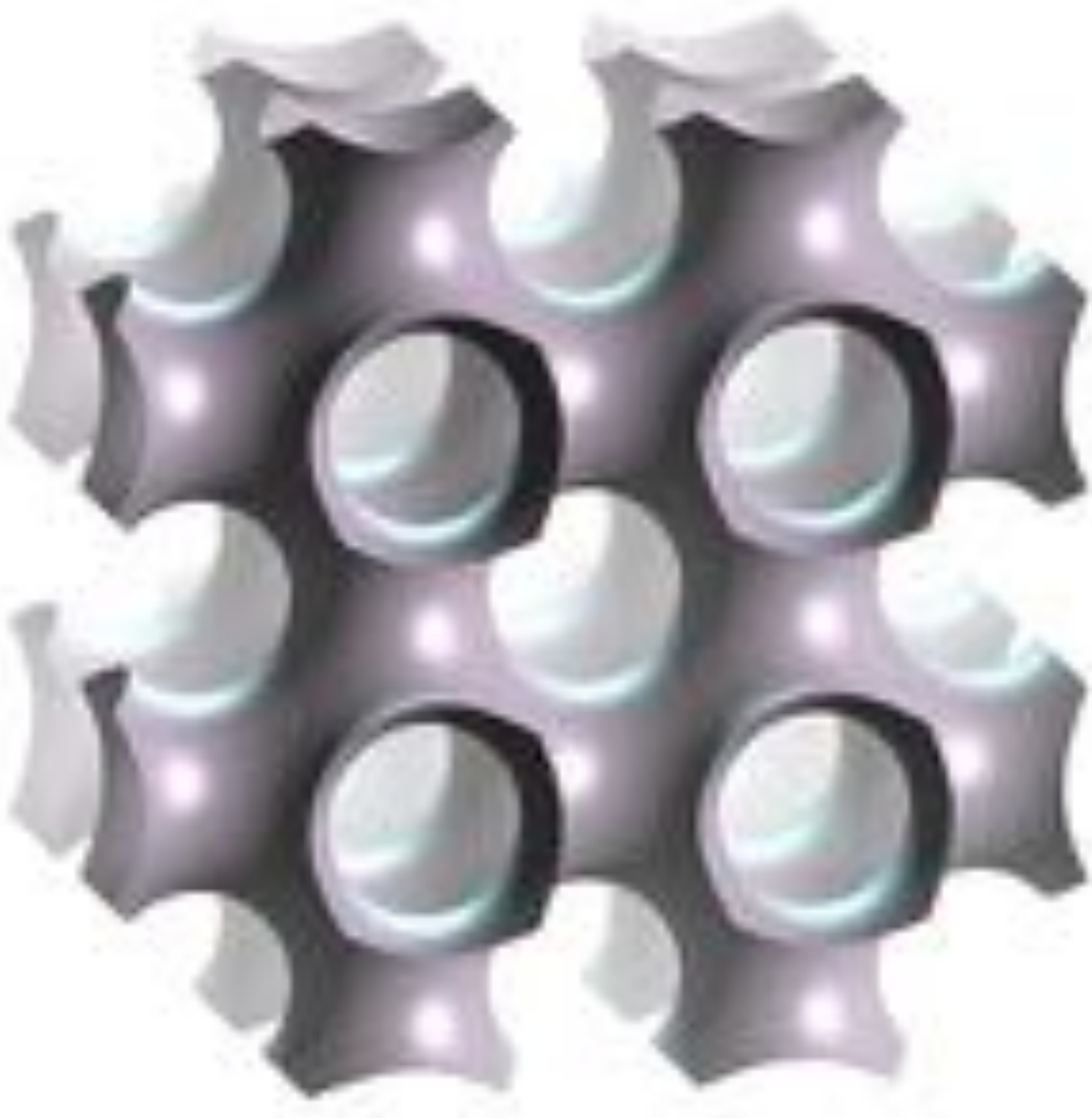
Faces-Edges+Vertices

32-64+32=0

Manifold *topology* sets the Euler characteristic χ



Infinite 3-periodic (crystal) surface:

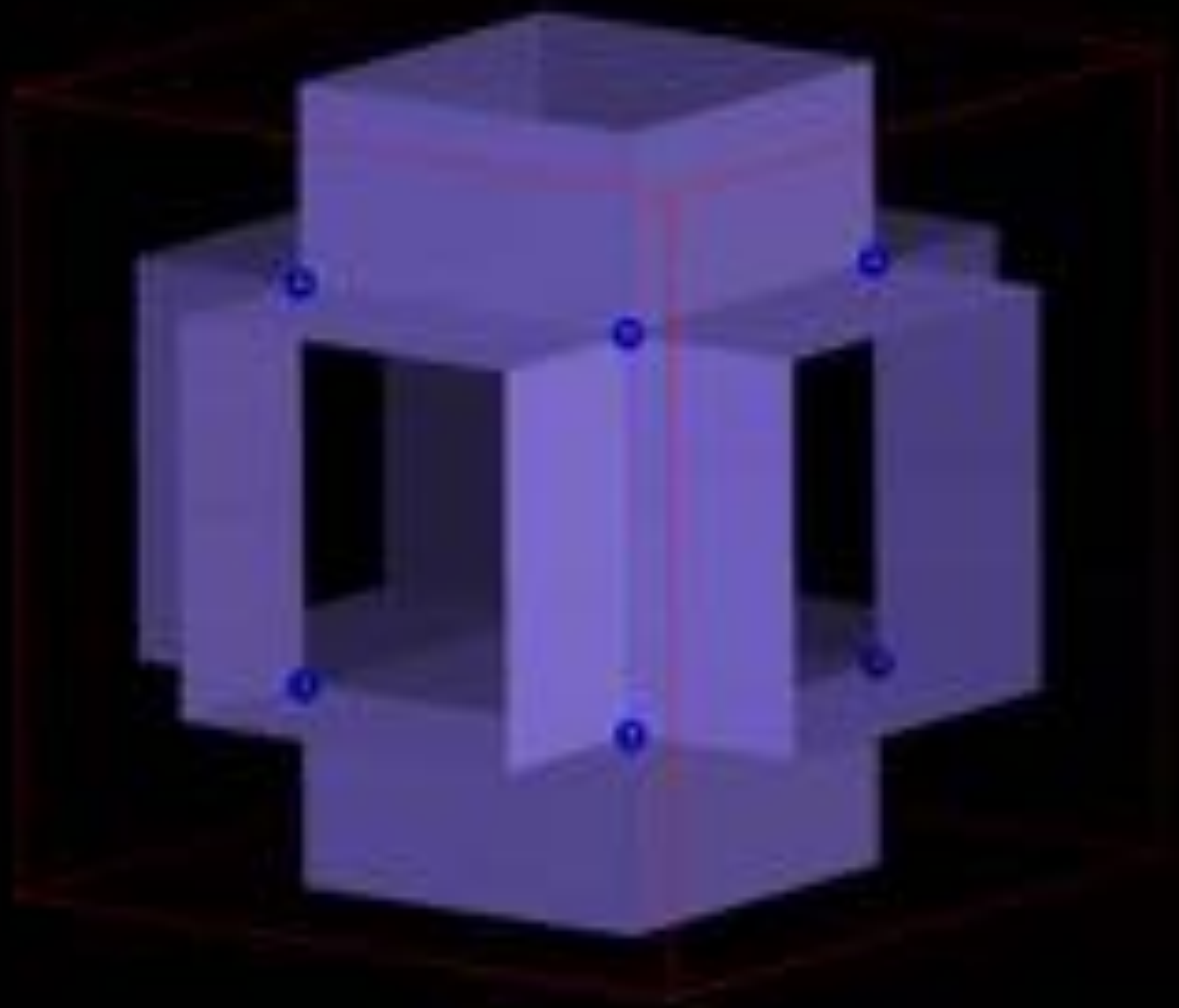


$$\chi = -\infty$$

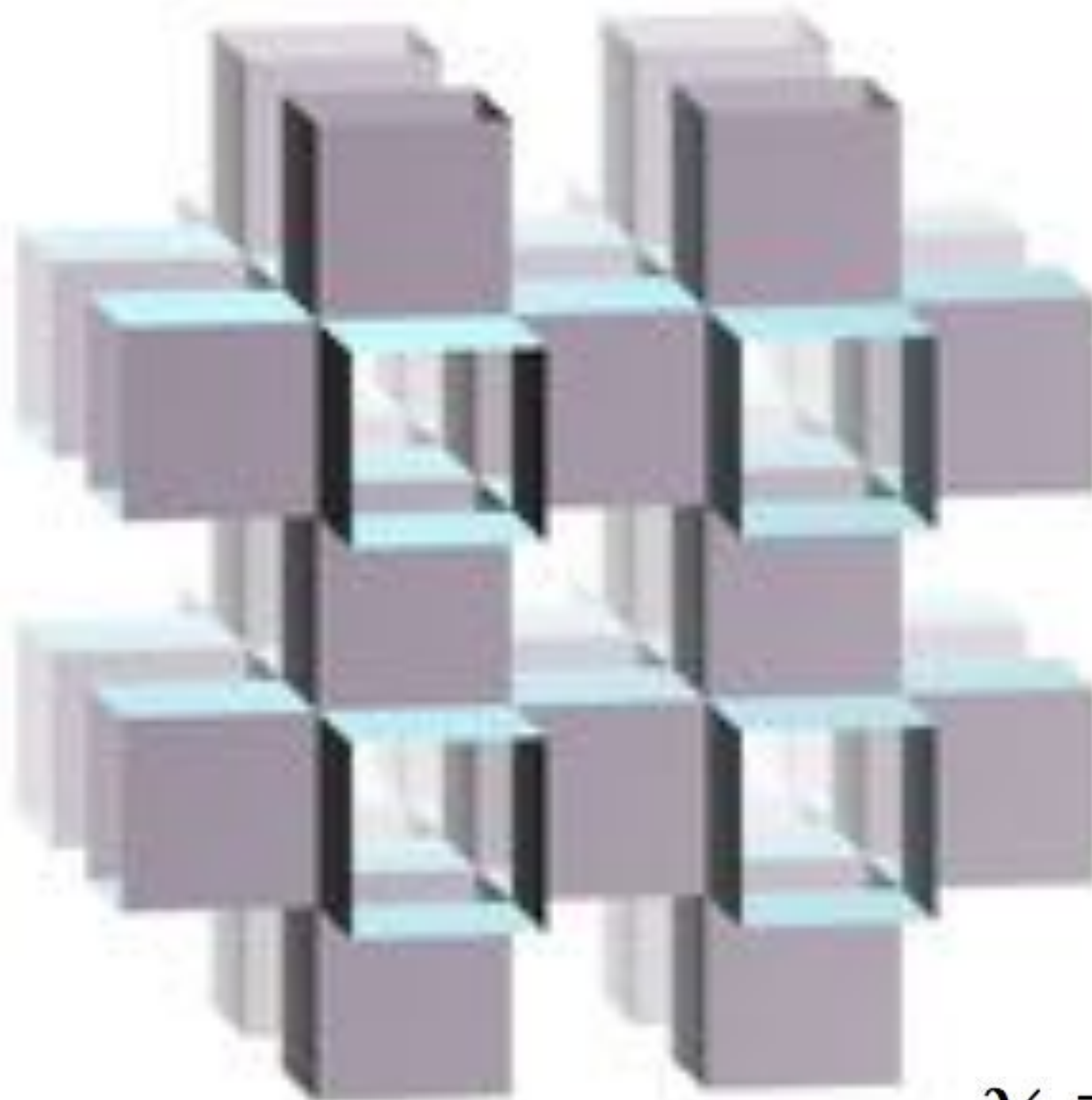
“infinite polyhedron”

$$\chi = -\infty$$

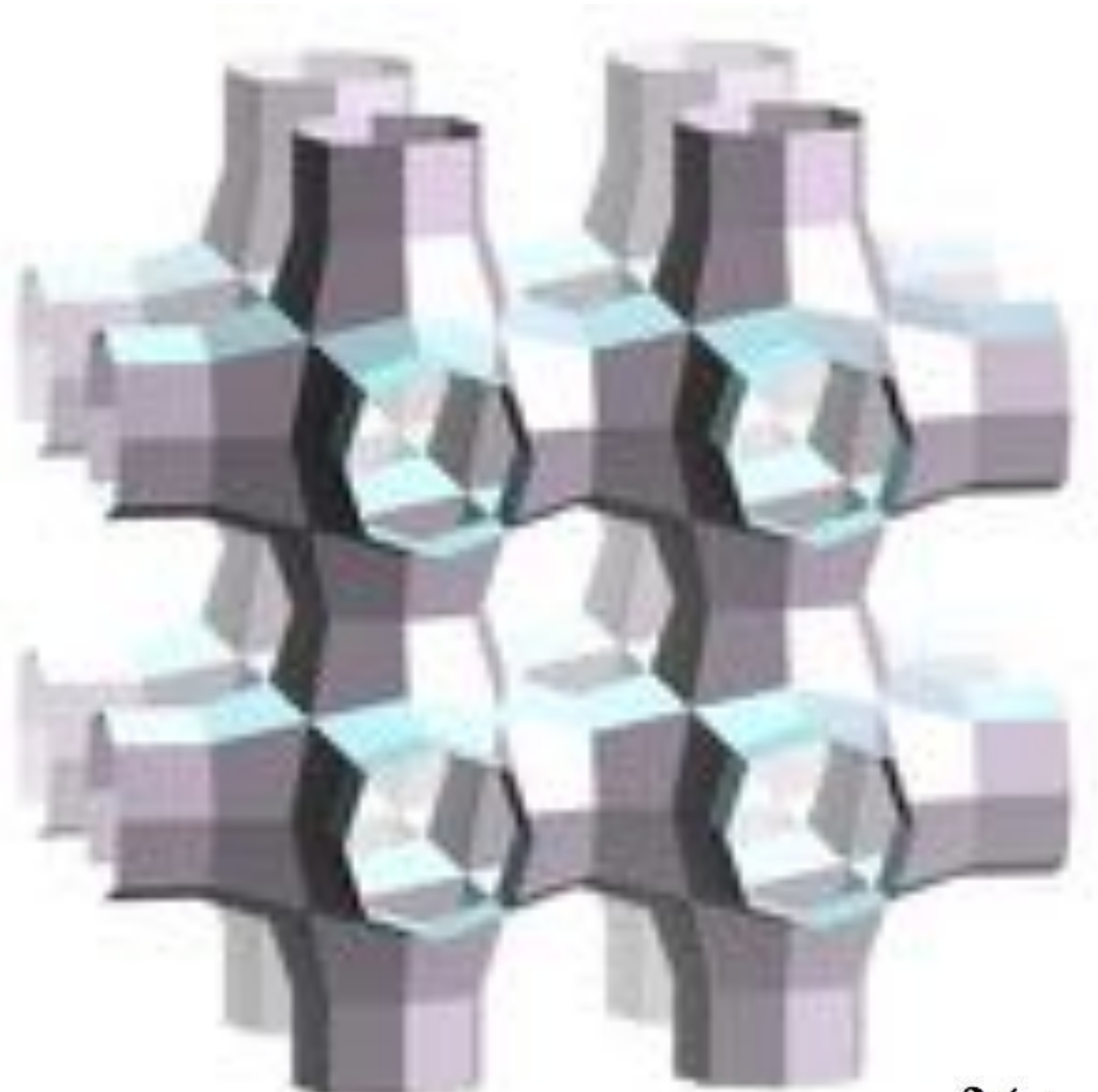
Per cubic unit cell:



cubic unit cell:	
	12
-edges	-24
vertices	8
Sum	-4



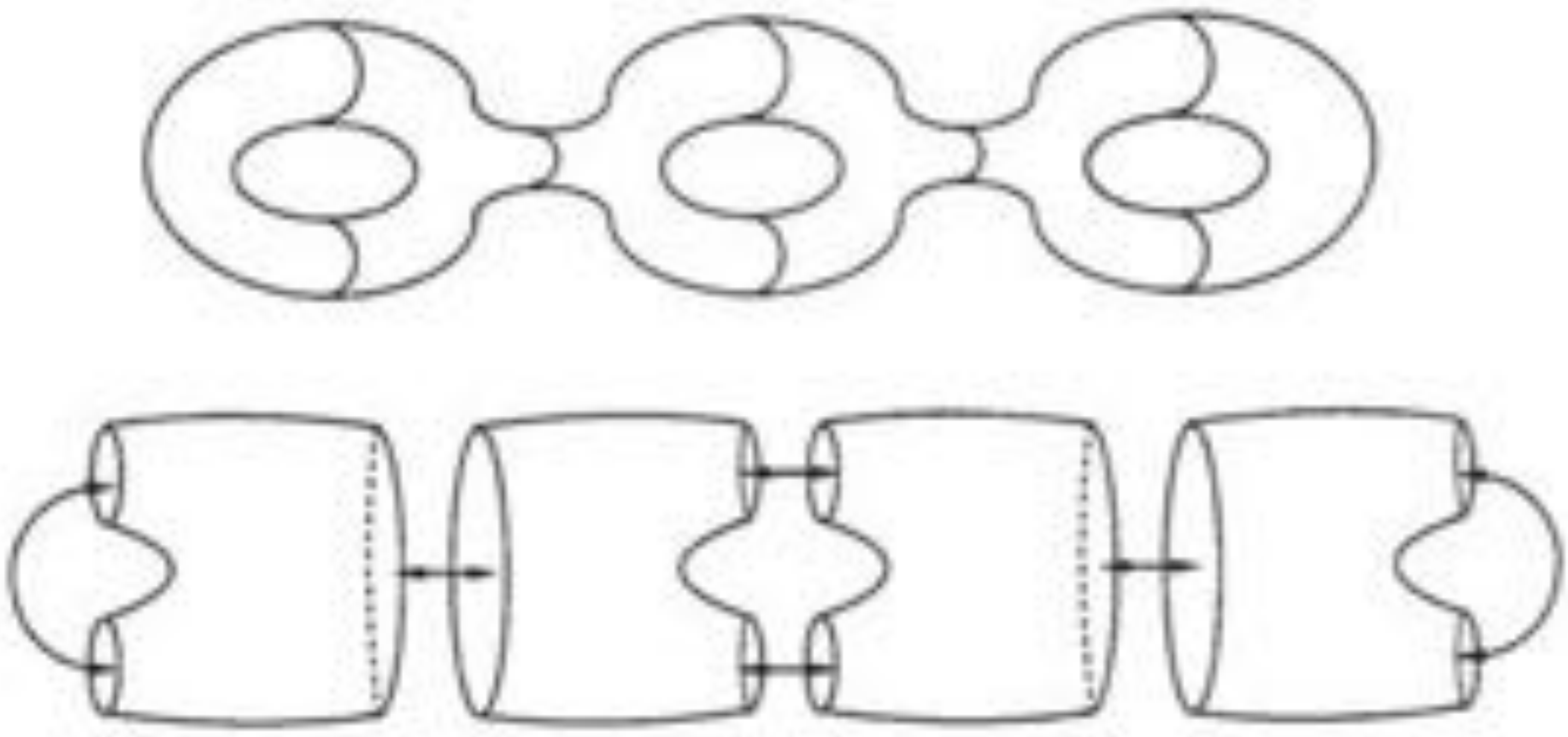
$\chi = -4$



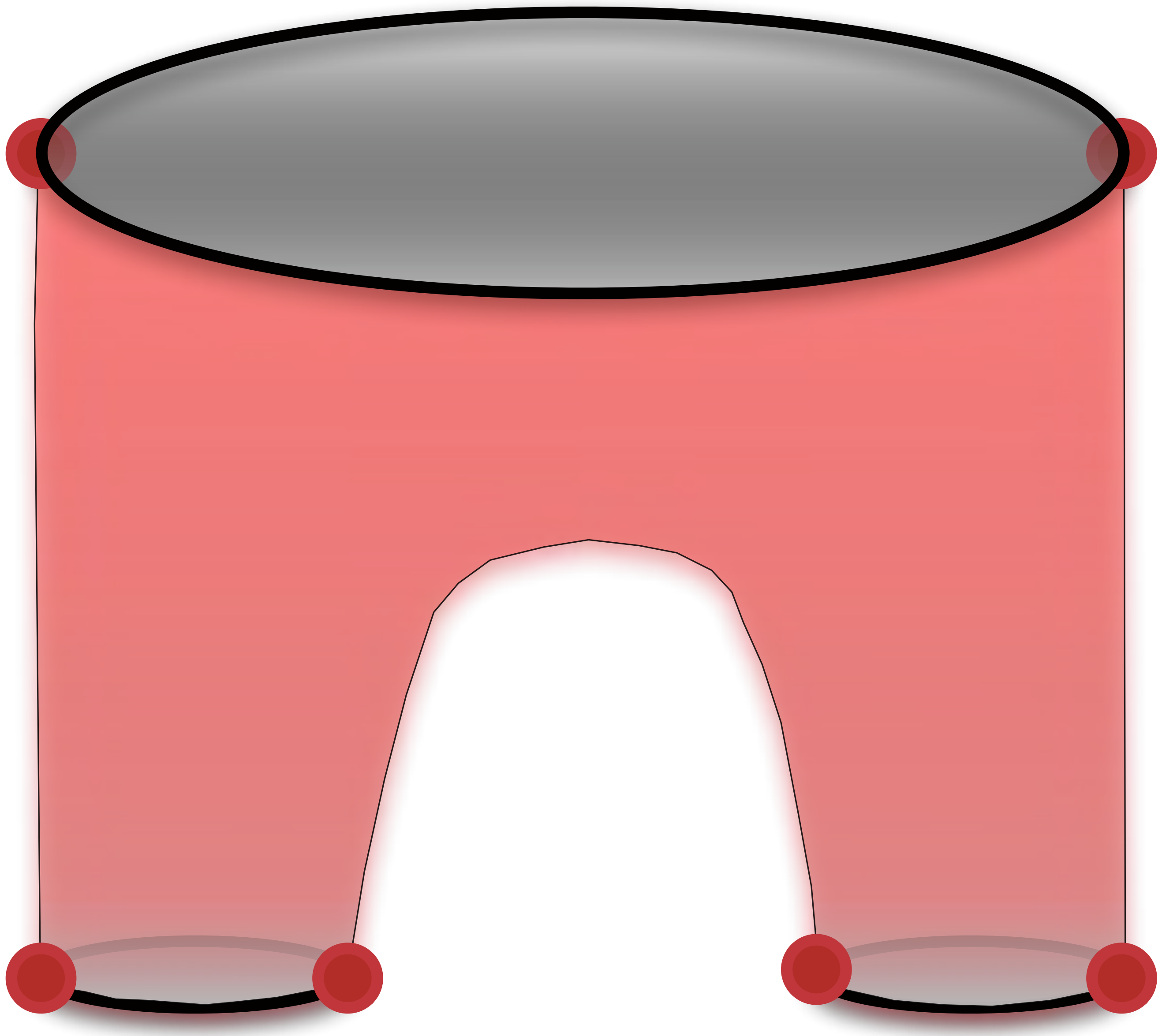
$\chi = -4$

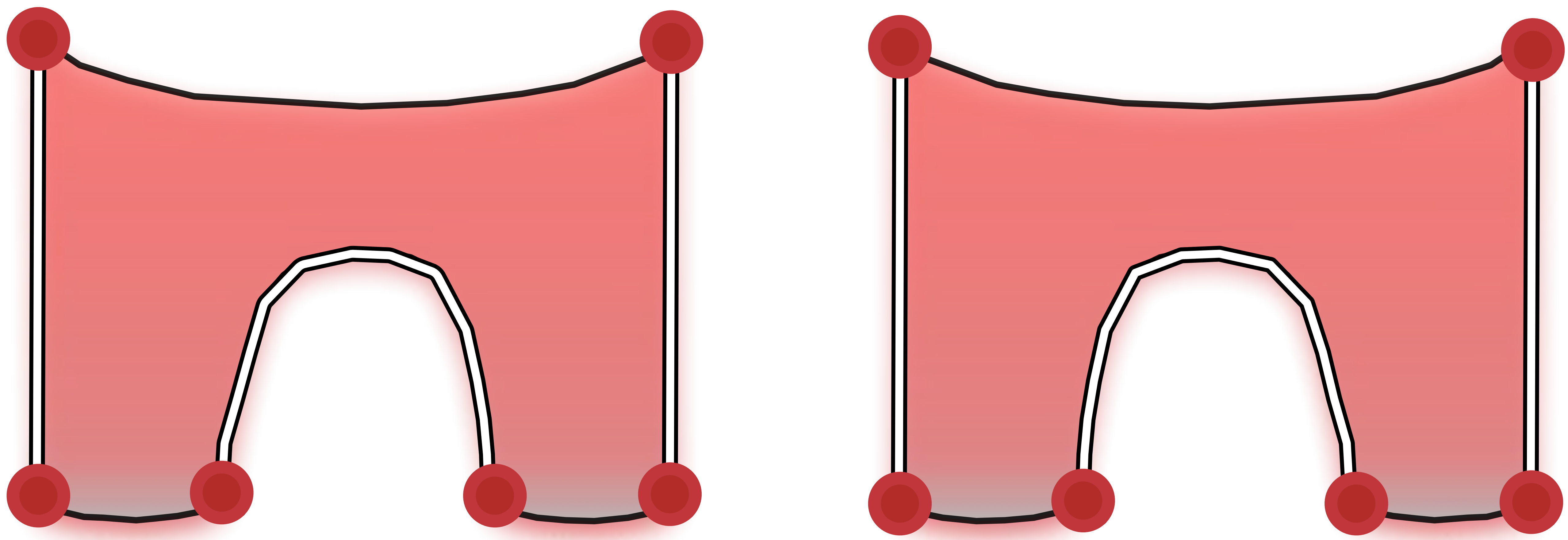
$\chi = -4$

Pants decomposition for a higher-genus manifold:



*Build all 'nice' manifolds from pairs of **pants**:*



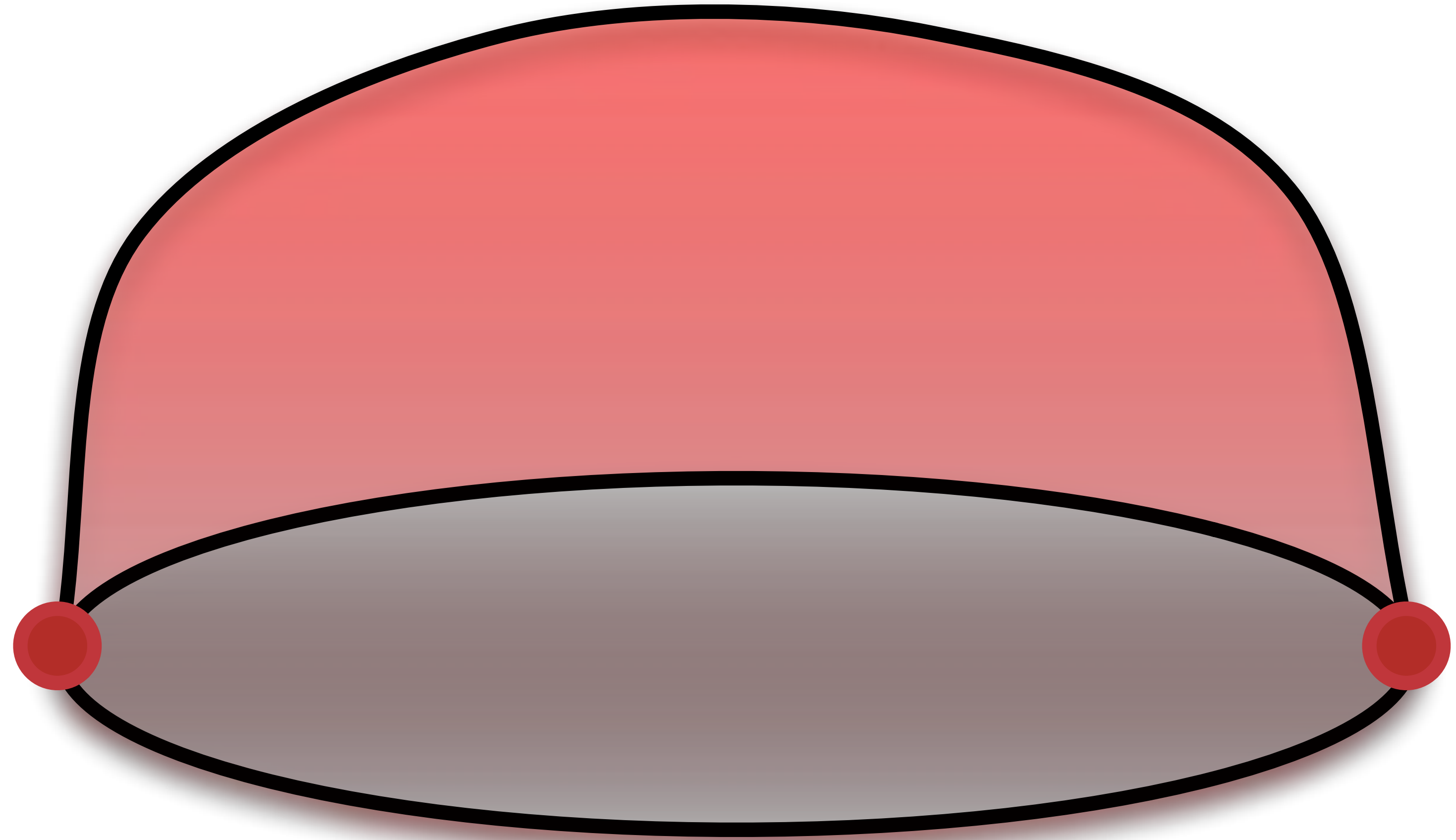


Euler characteristic of pants:

V=12/2=6 ; E = 6+6/2=9; F=2

$$\boxed{\chi_{pants} = -1}$$

.....and caps:



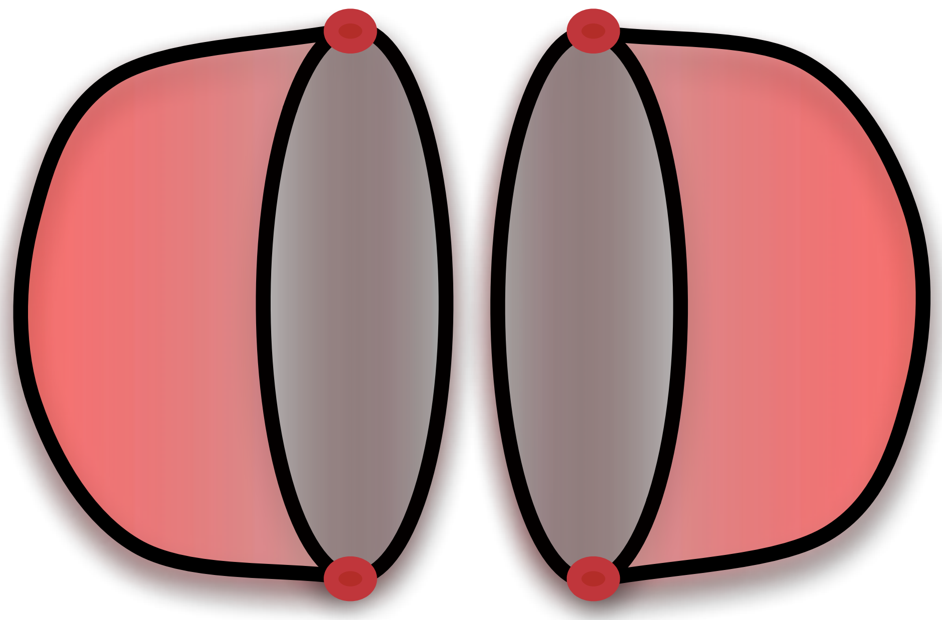
V=2 ; E = 3; F=2

$$\boxed{\chi_{cap} = 1}$$

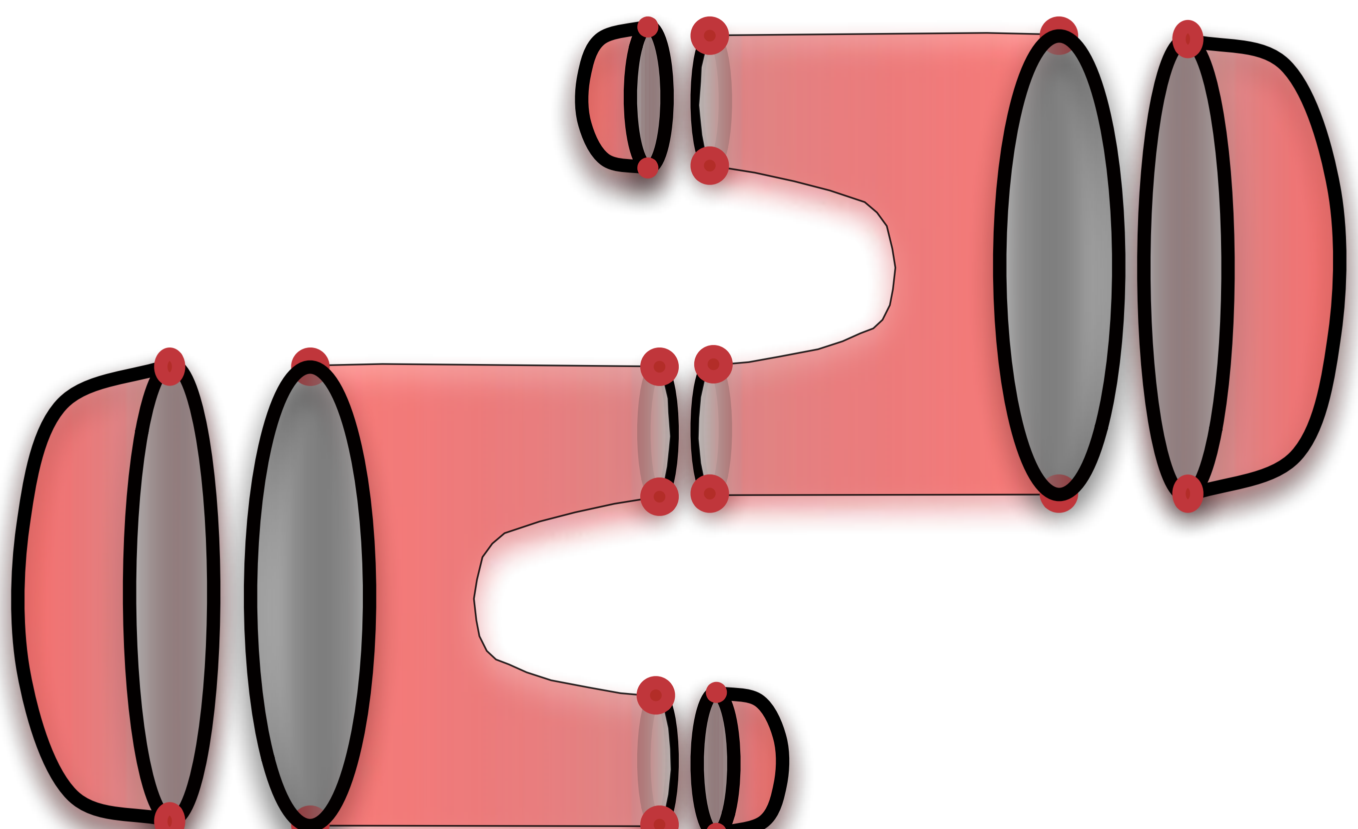
Euler characteristic of 'nice' manifolds:

$$\chi = \#\ pants.\chi_{pants} + \#caps.\chi_{cap}$$

$$\boxed{\chi = (\#\ caps - \#\ pants)}$$

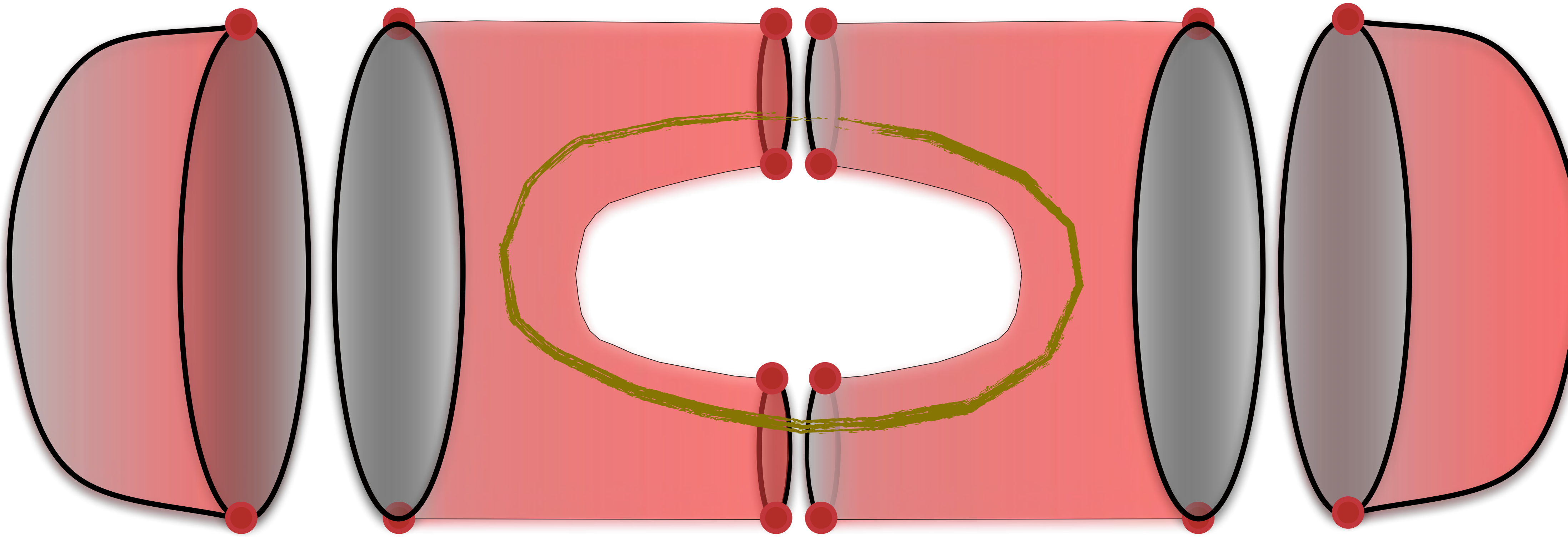


2 caps



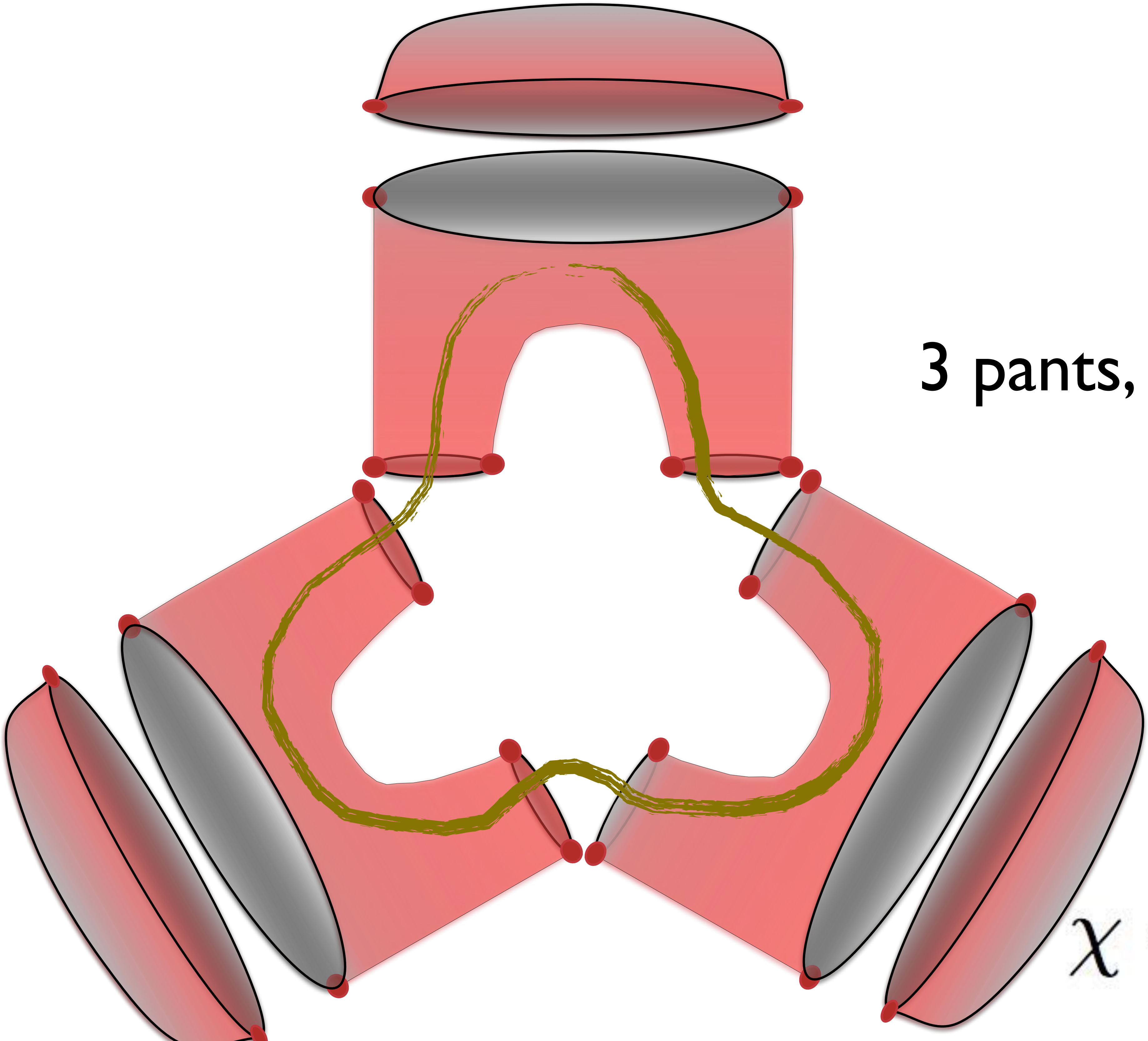
2 pants, 4 caps

$\chi = 2$



2 pants, 2 caps

$\chi = 0$



3 pants, 3 caps

$\chi = 0$

GENUS of 'nice' manifolds, "g":

$$\chi = (\#\ caps - \#\ pants)$$

$$g = 1 - \frac{\chi}{2}$$

$$\boxed{g = 1 + \frac{\#pants - \#\ caps}{2}}$$

Manifold *topology* sets the Euler characteristic χ and genus

$\chi = 2$

genus=0

$\chi = 0$

genus=1

$\chi = -2$

genus=2

$\chi = -4$

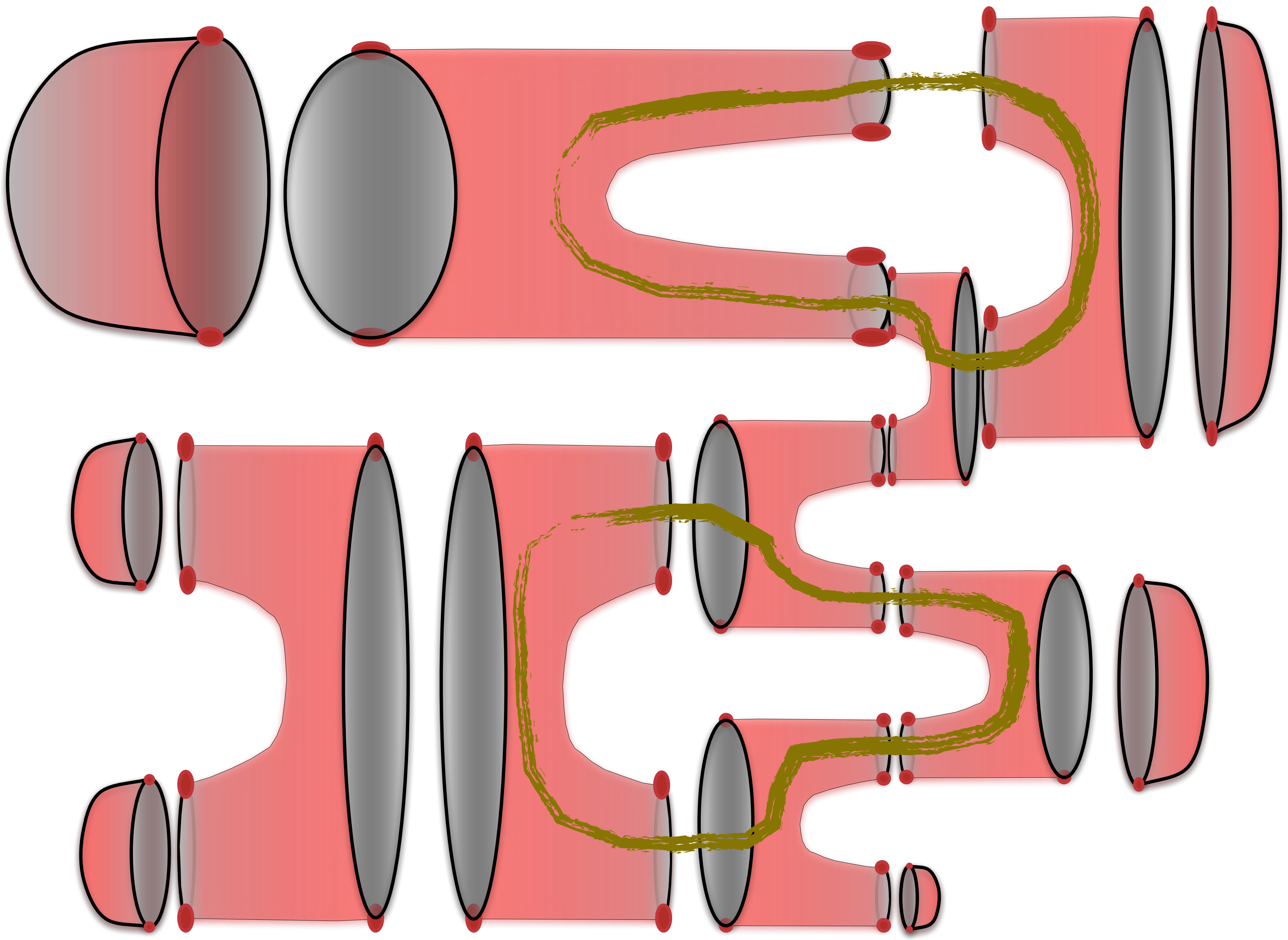
genus=3

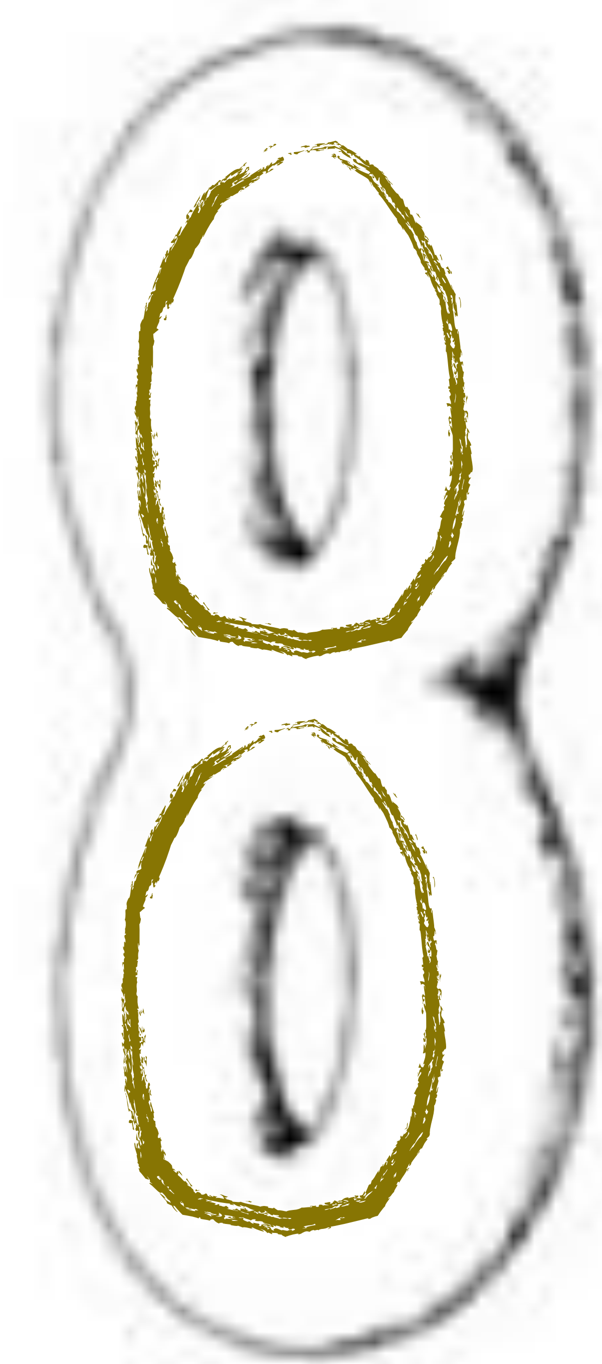
Euler characteristic describes manifold topology.

$$F - E + V = \chi$$

Genus describes number of loops or “handles”

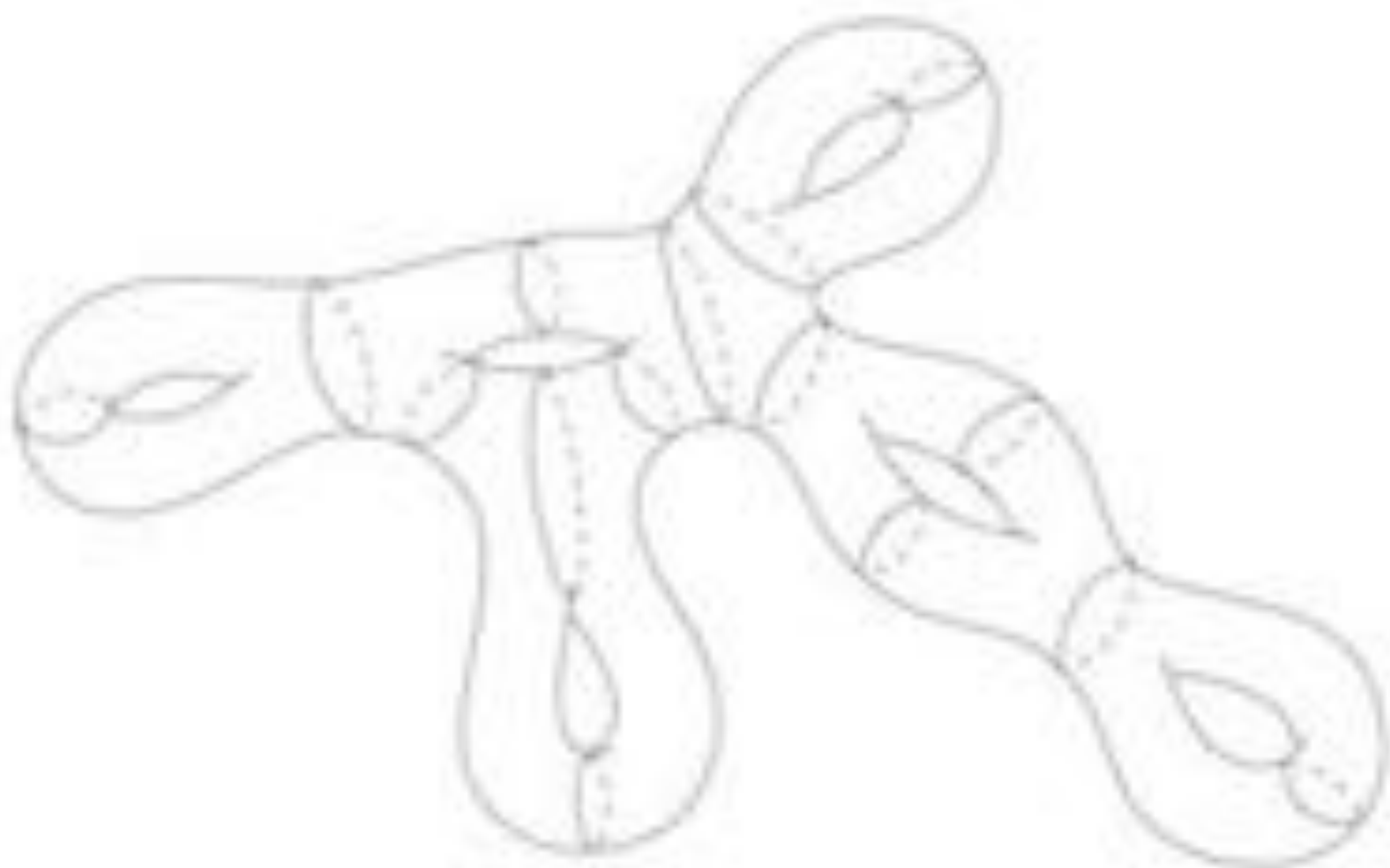
$$g = 1 - \frac{\chi}{2}$$





8 pants, 6 caps: $\chi = -2$, genus=2

In fact, any closed orientable manifold can be cut into pants only...



Question: *What is minimum # cuts needed to build an unbounded genus-g manifold?*

$$g = 1 + \frac{\#pants - \#\ caps}{2}$$

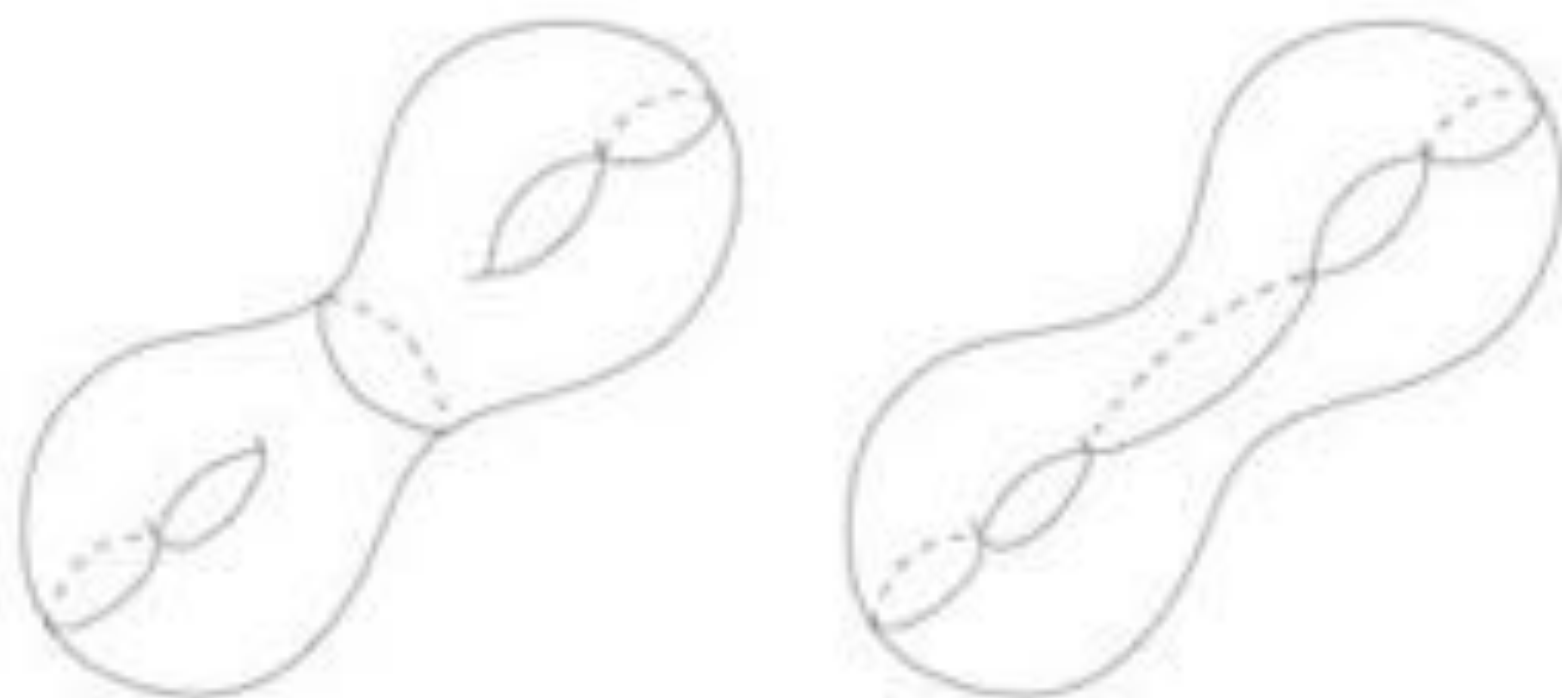
So nice manifold of genus g has:

$$2g - 2\ pairs\ of\ pants$$

each pants has 3/2 closed loops.....

So, 3(g-1) cuts needed

Theorem 3-4. *A hyperbolic Riemann surface R of genus g always contains a loop system of $3g-3$ disjoint simple closed geodesics. Regardless of which loop system we choose, cutting R along the geodesics in the system always decomposes R into $2g-2$ pairs of pants.*



e.g. genus=2, 2 pants.

REMARK: Each loop is independent, so 3(g-1) loop-lengths

add Dehn twist angle ($2\pi N$):

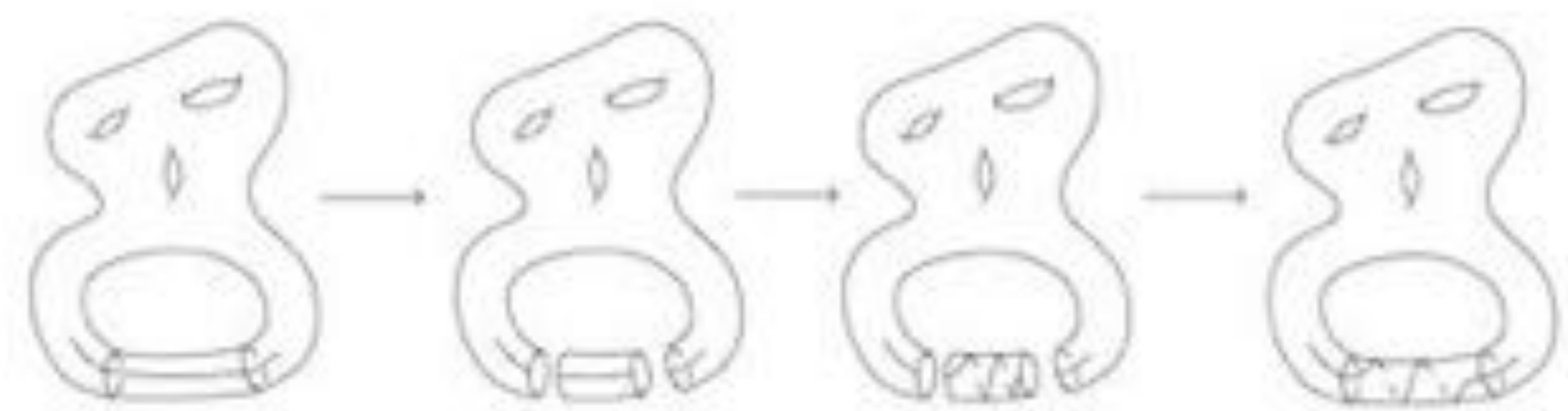


Figure 4-1. Dehn twisting.

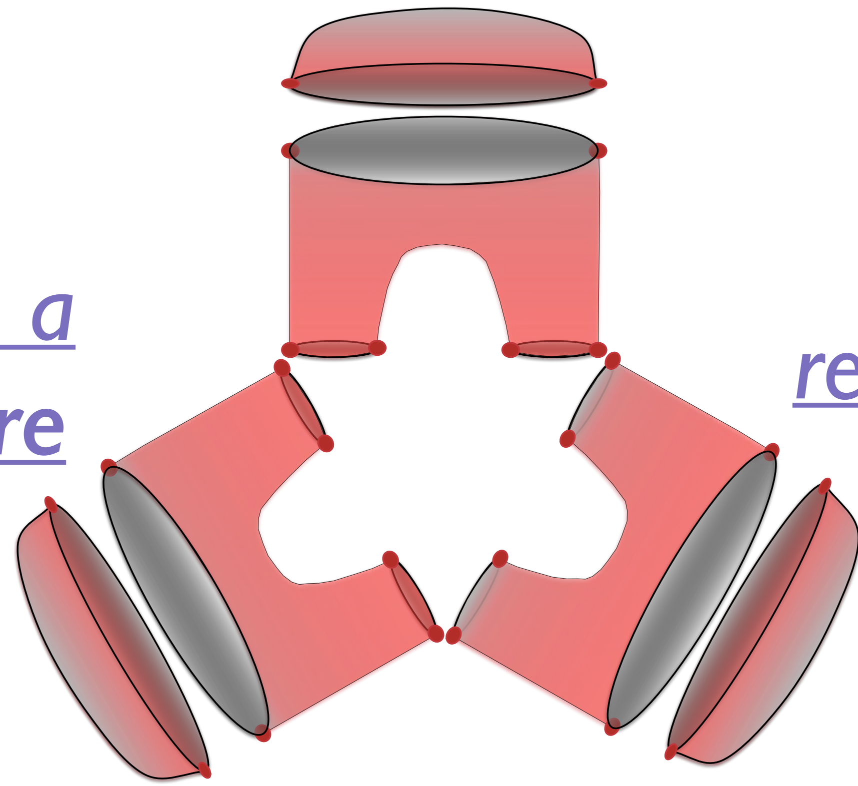
....3(g-1) angles

<u>Uniform curvature manifold defined by 6(g-1) parameters</u>

(Fenchel-Nielsen coordinates)

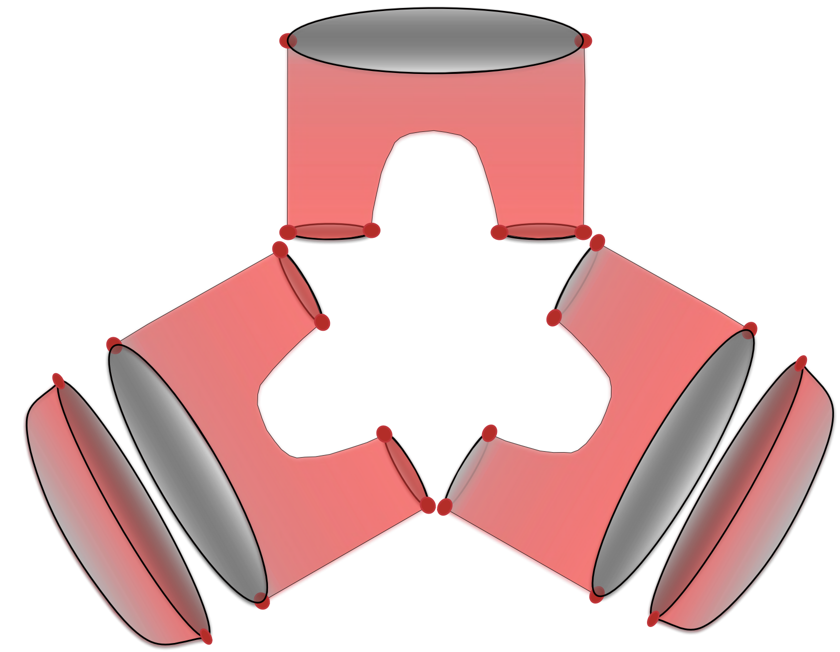
‘nice’ manifolds with punctures:

to make a puncture



$\chi = 0$

remove a cap



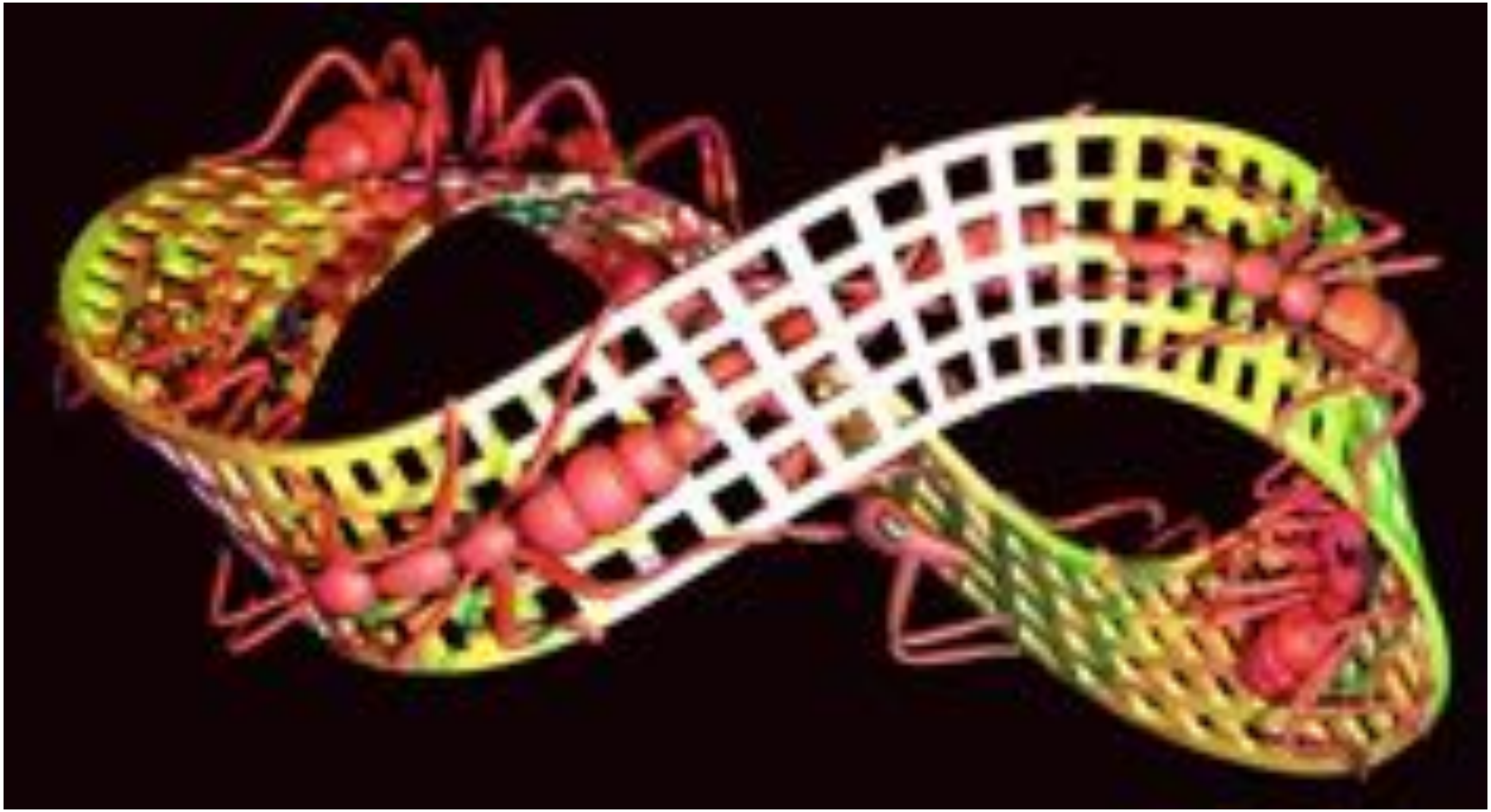
$\chi = -1$

$$\chi = \#\,pants.\chi_{pants} + \#caps.\chi_{cap} + \#holes.\chi_{holes}$$

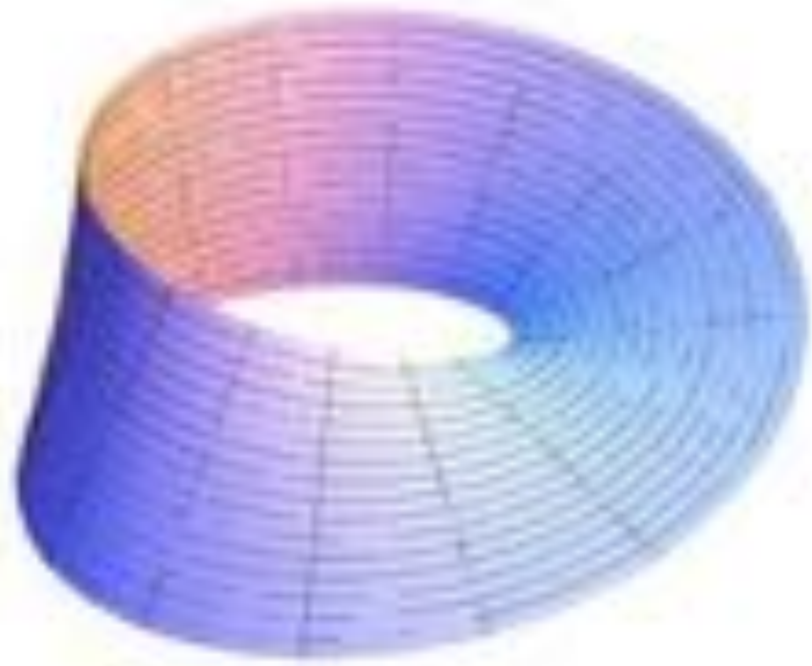
$$= \#caps - \#\,pants - \#\,holes$$

- orientable (nice) vs. non-orientable (nasty) surfaces

Möbius strip

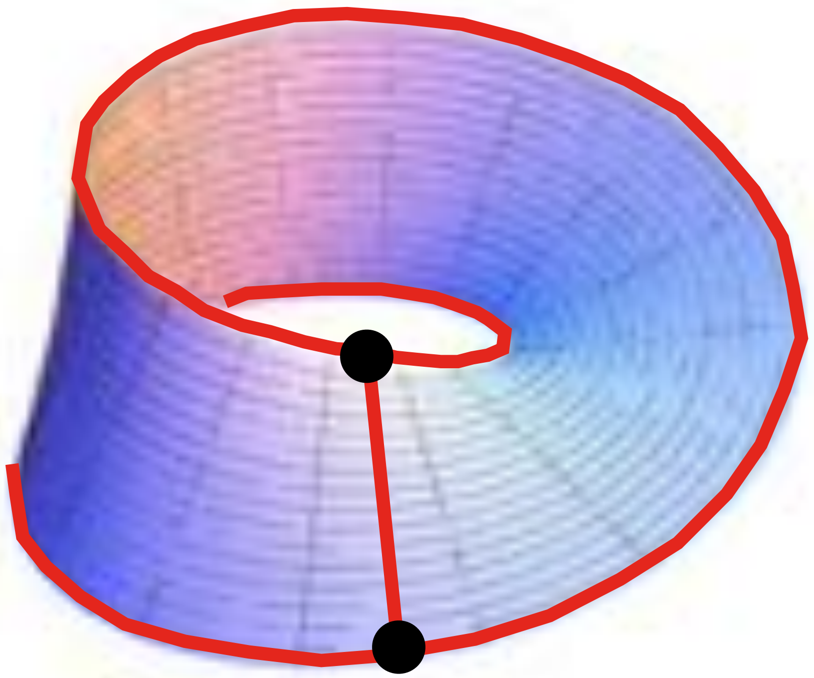


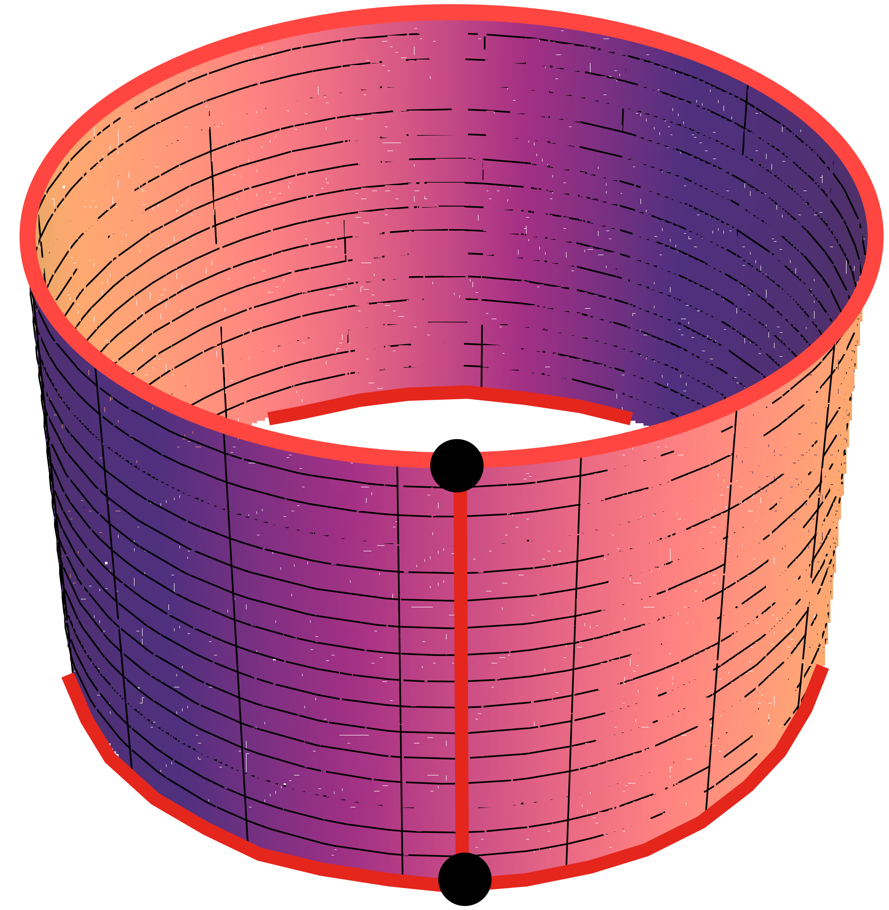
http://www.cs.technion.ac.il/~gershon/EscherForReal/MoebiusAnt.gif



1 Face
3 Edges
2 Vertices

$$\boxed{\text{F-E+V=0}}$$



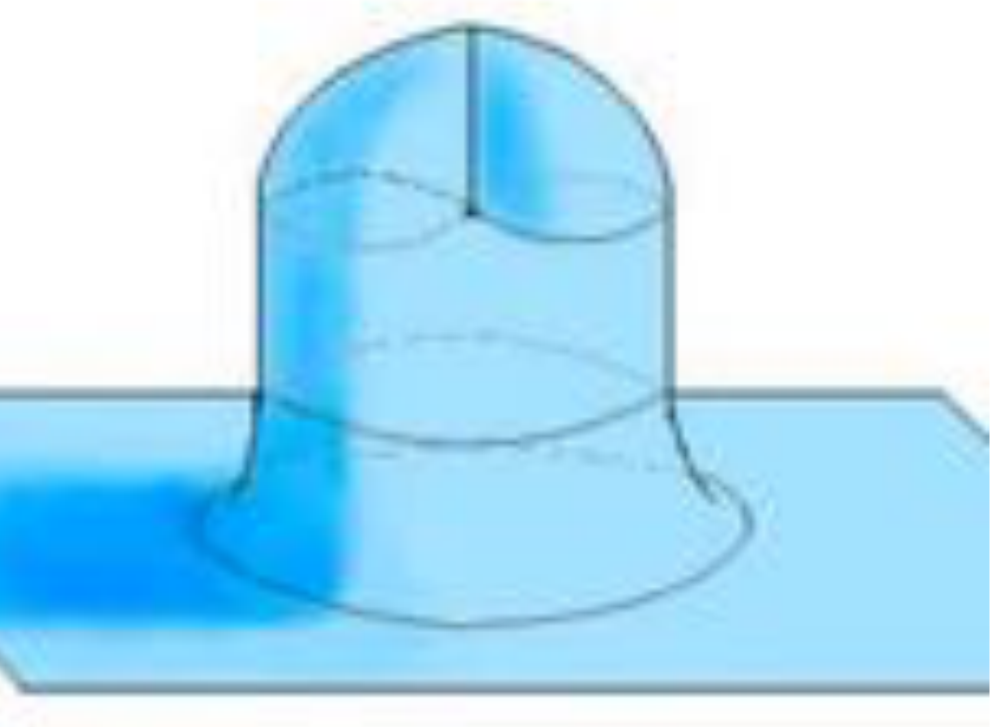


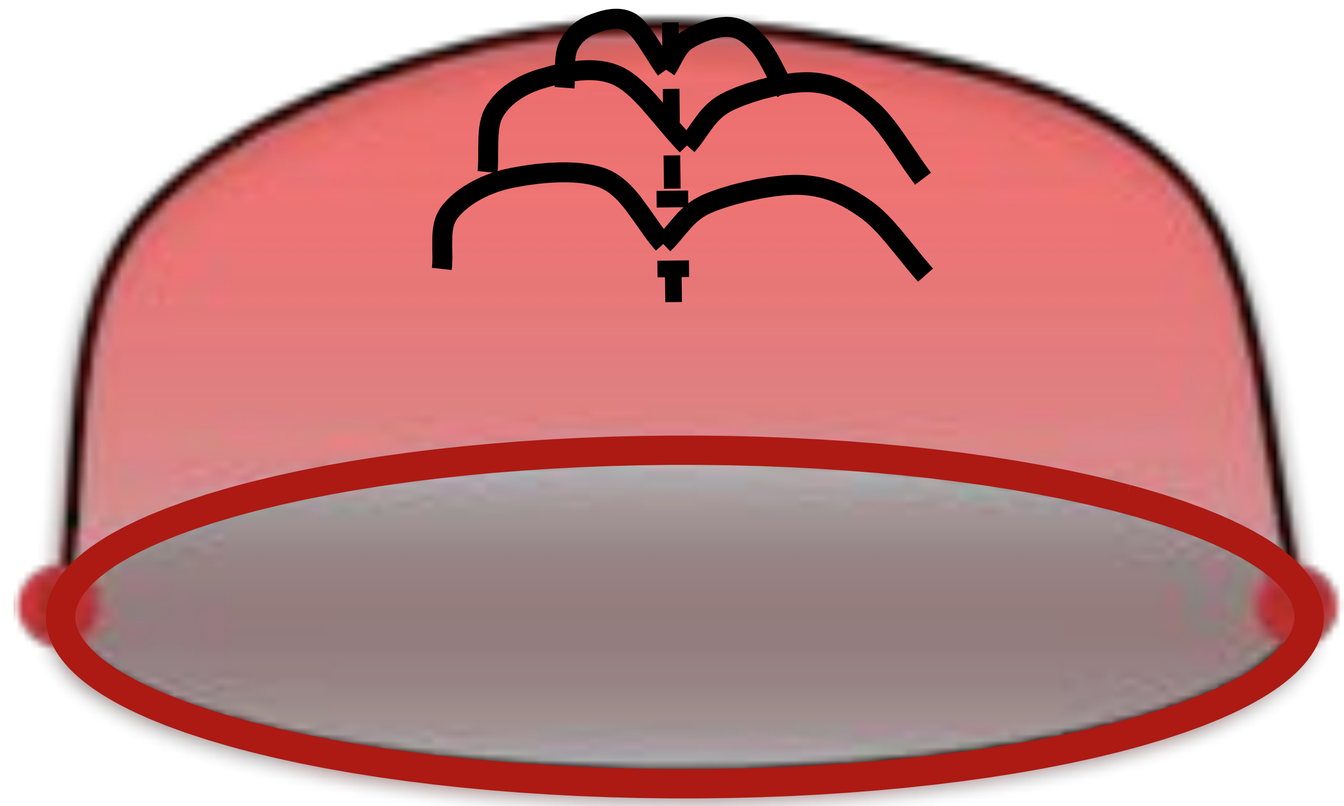
1 Face
3 Edges
2 Vertices

$$\boxed{\text{F-E+V=0}}$$

$$\boxed{\chi = 0}$$

Nasty manifolds can have same Euler characteristic as nice manifolds, but topologically different



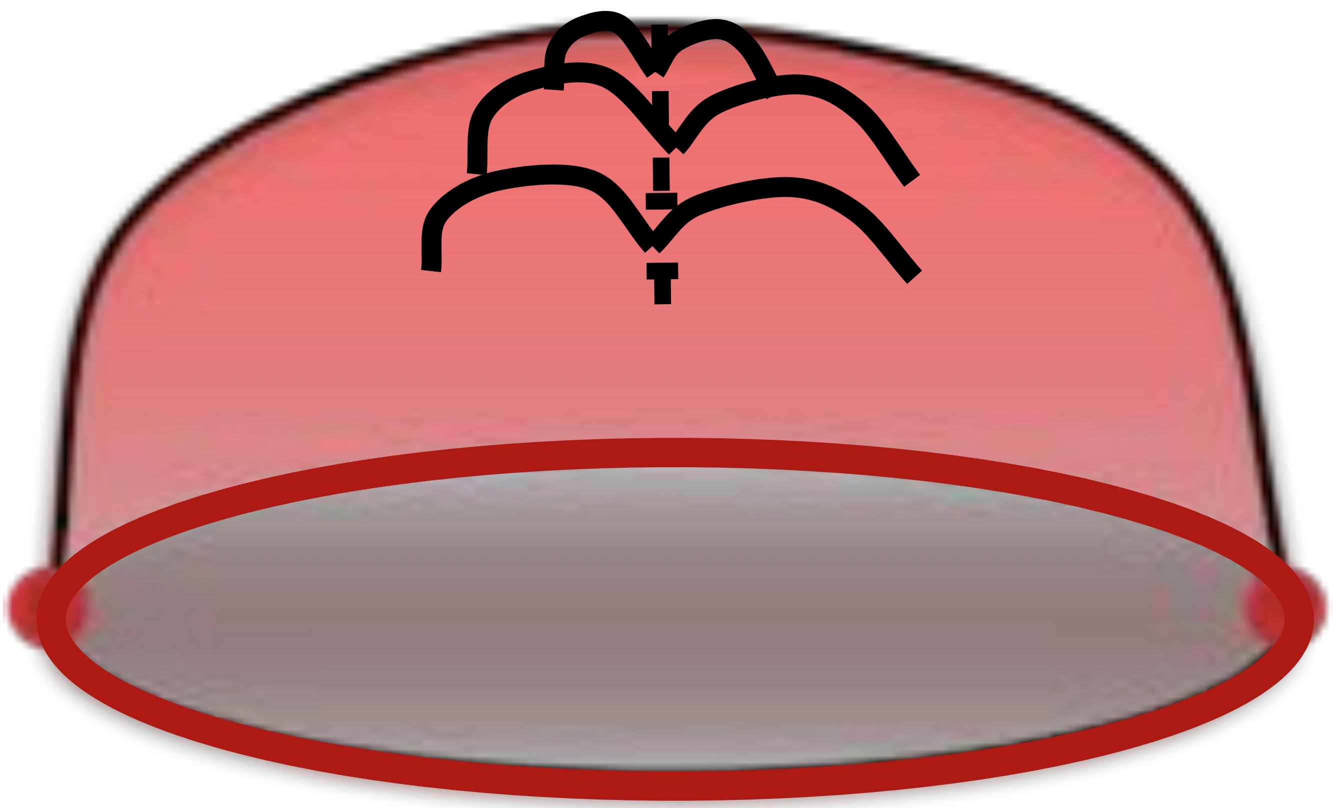


(vertices and edges are shared with adjacent modules):

V=2/2=1 ; E = 2/2+1=2; F=1

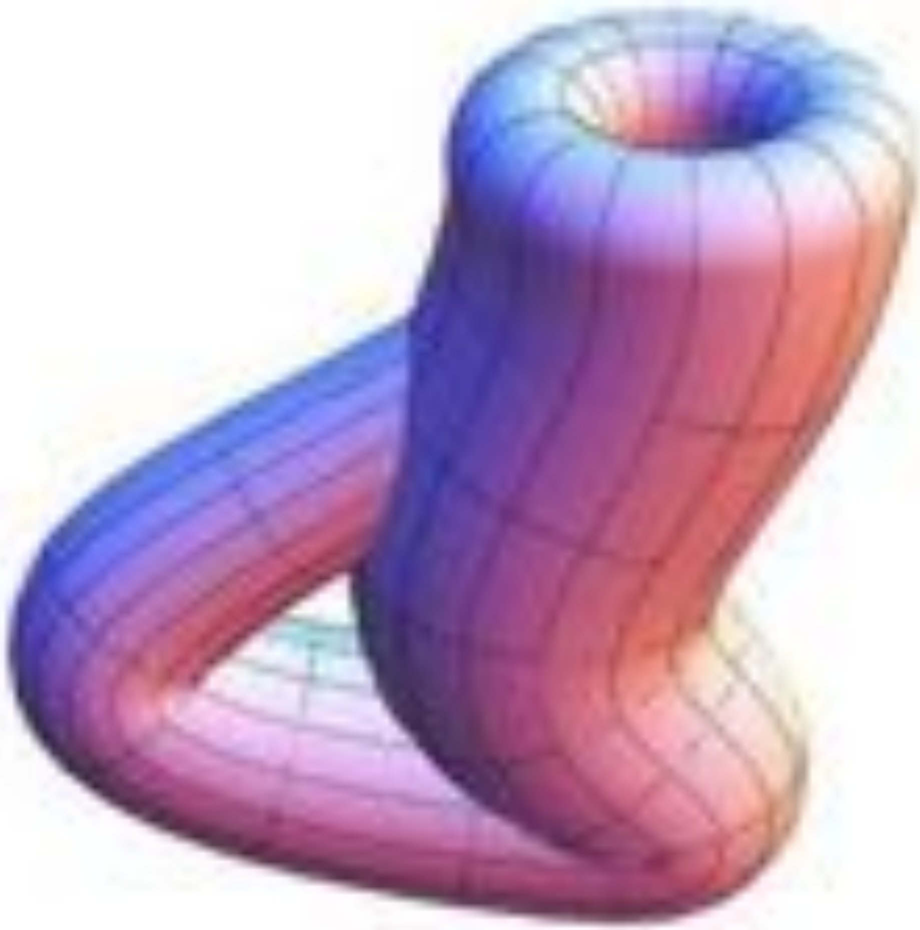
$$\boxed{\chi_{xcap} = 0}$$

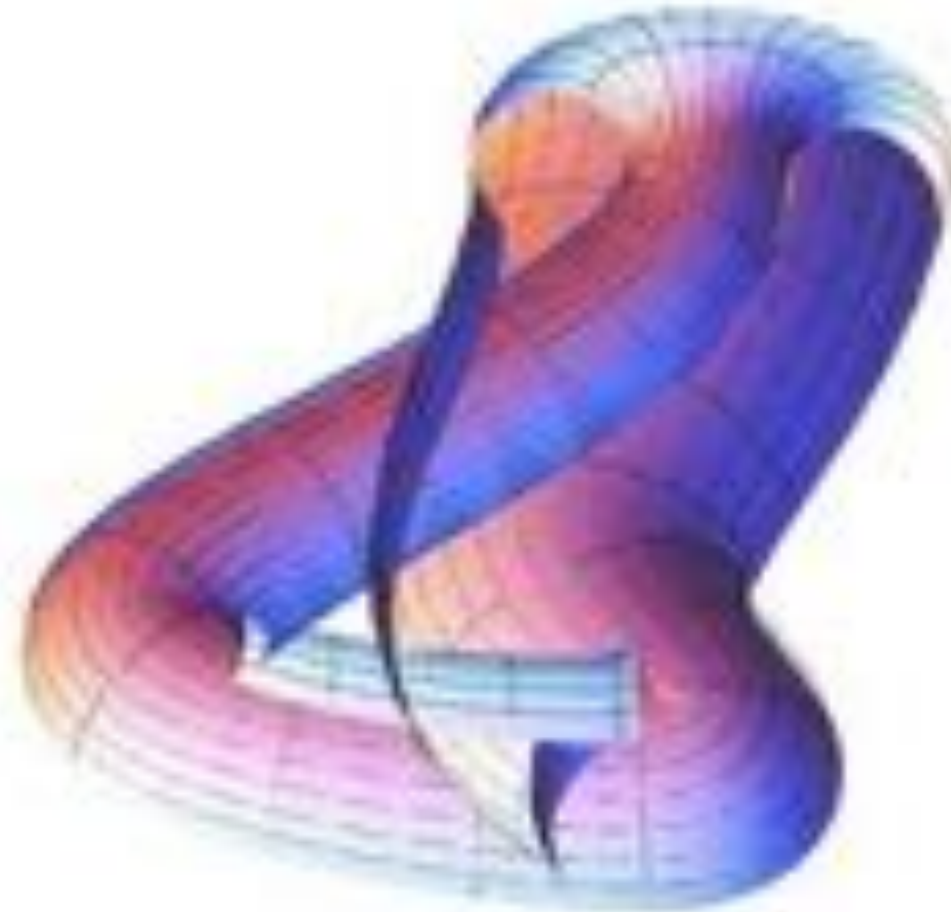
Moebius band is topologically identical to cross-cap!

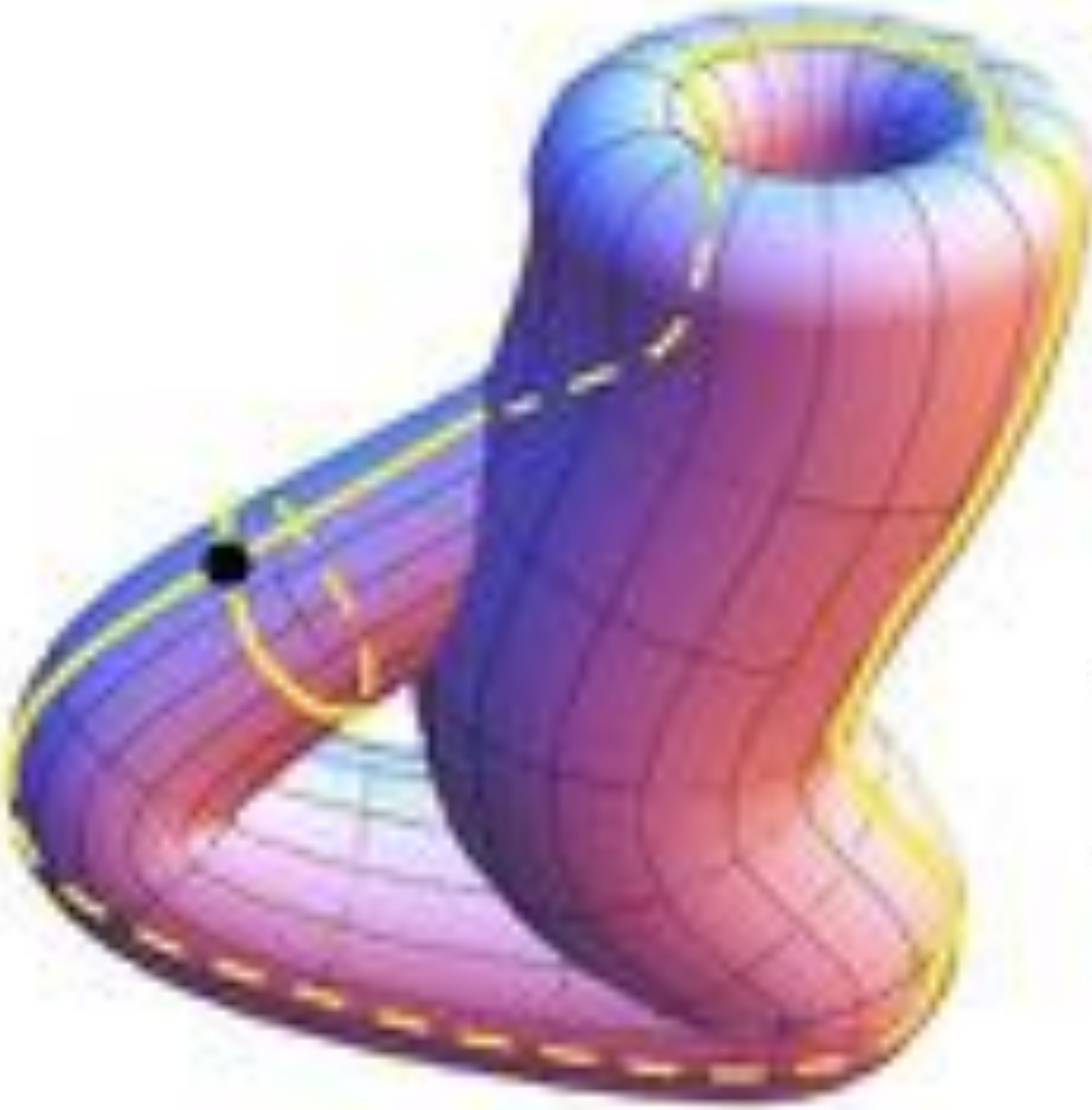


$$\boxed{\chi_{xcap} = 0}$$

another nasty manifold: Klein “bottle”



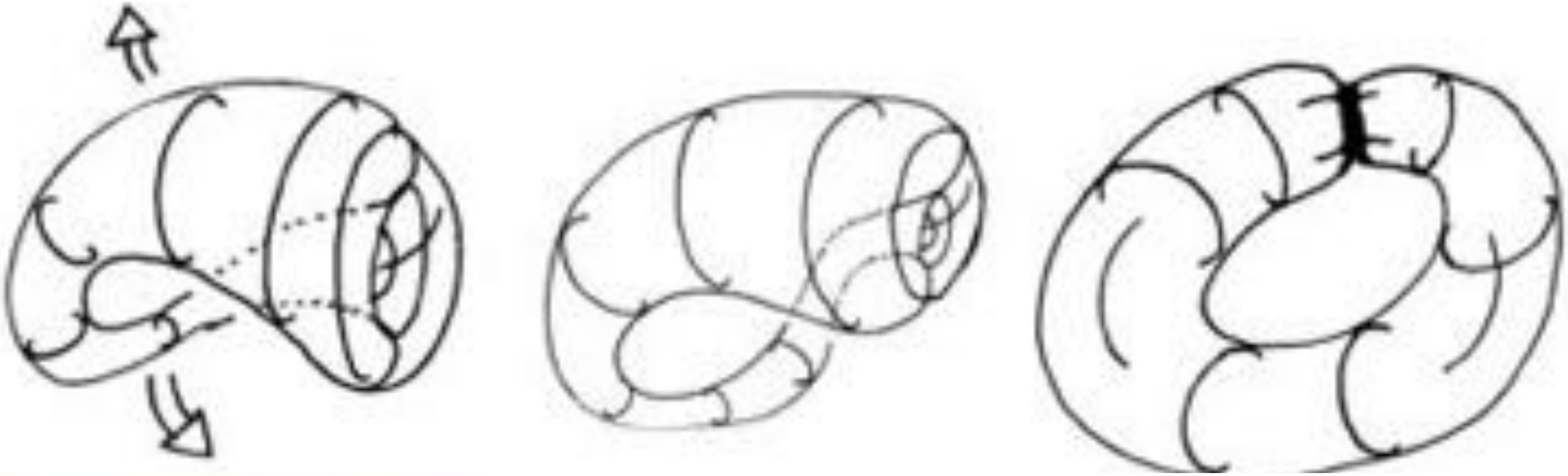


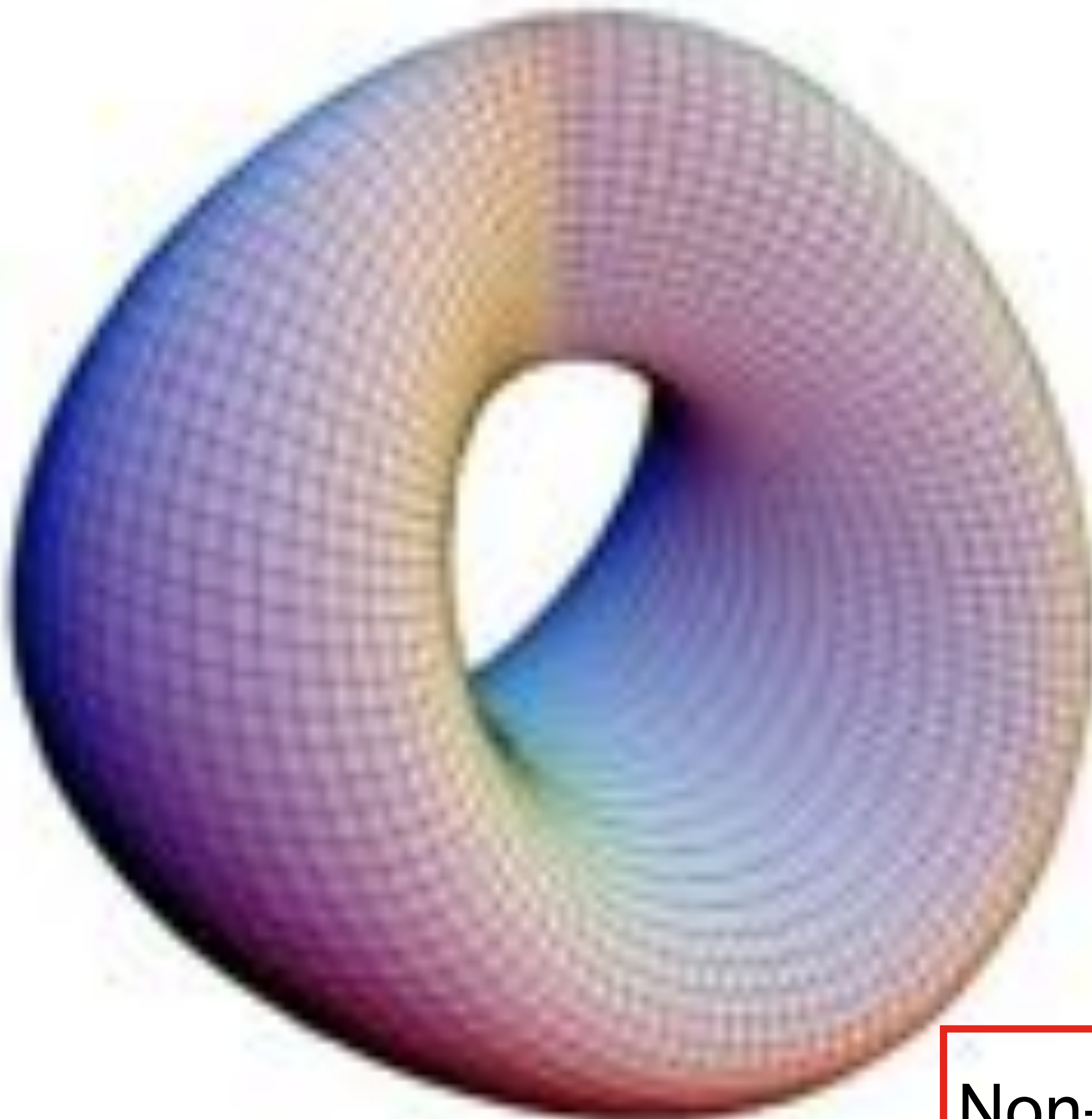


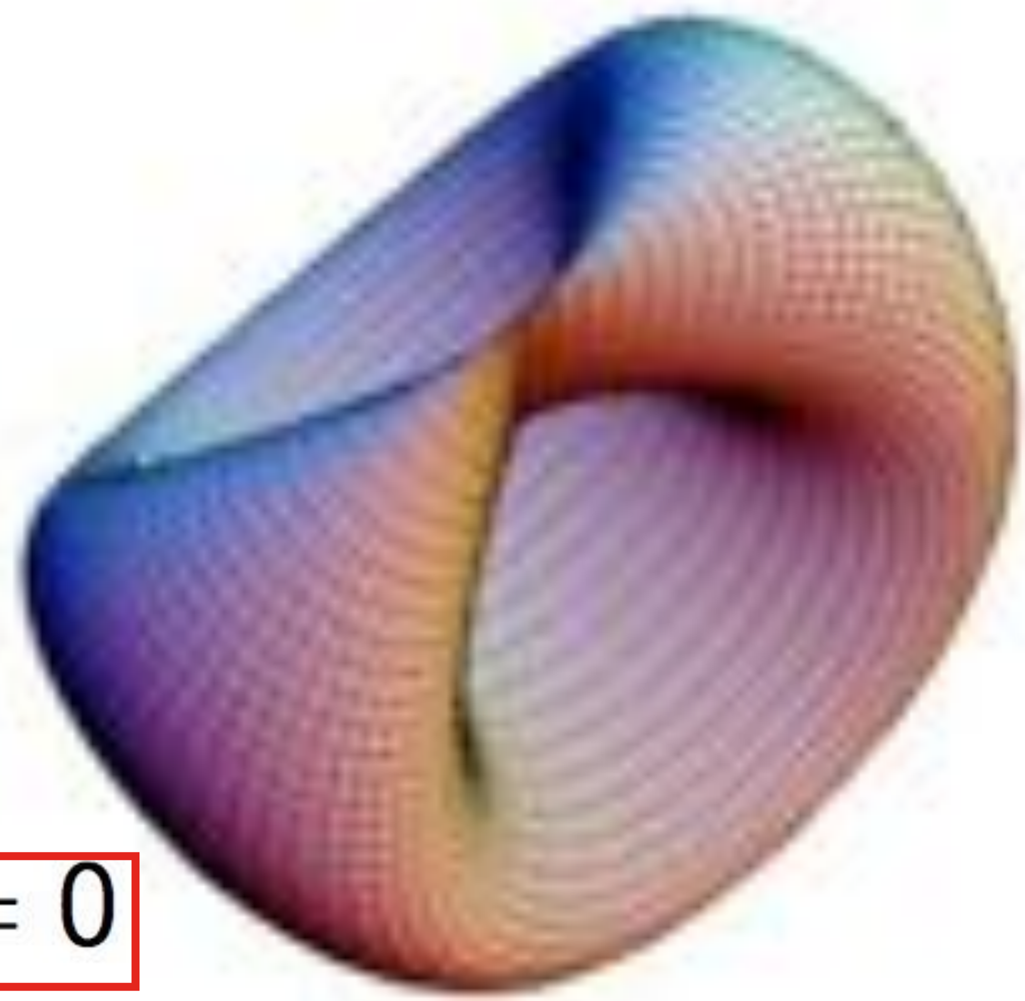
1 Face
2 Edges
1 Vertex

$$\boxed{\text{F-E+V=0}}$$

$$\boxed{\chi = 0}$$



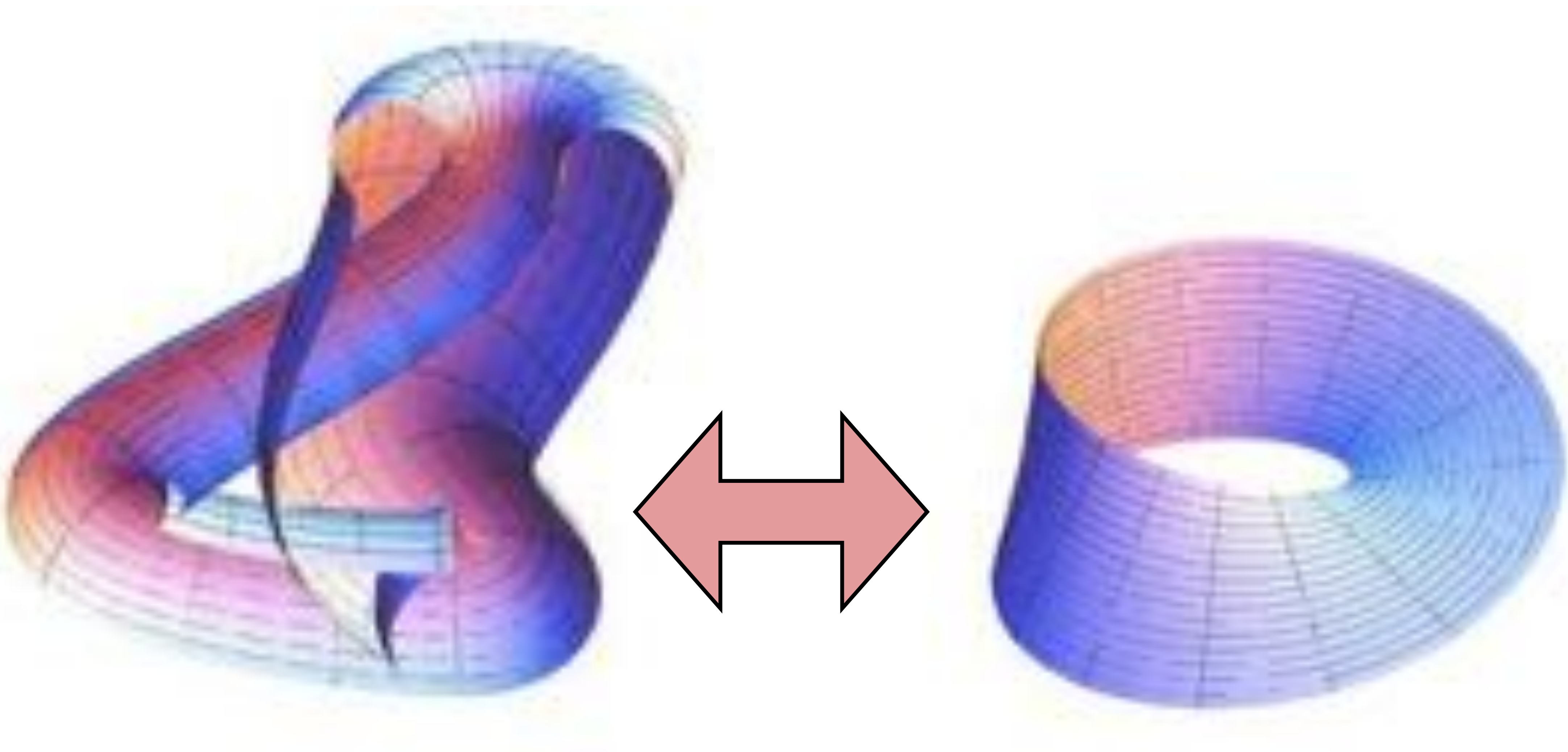




$\boxed{\chi = 0}$

Non-oriented genus =1

A half-Klein bottle is a Möbius band



Zip two Möbius bands together along boundary: form a Klein bottle!

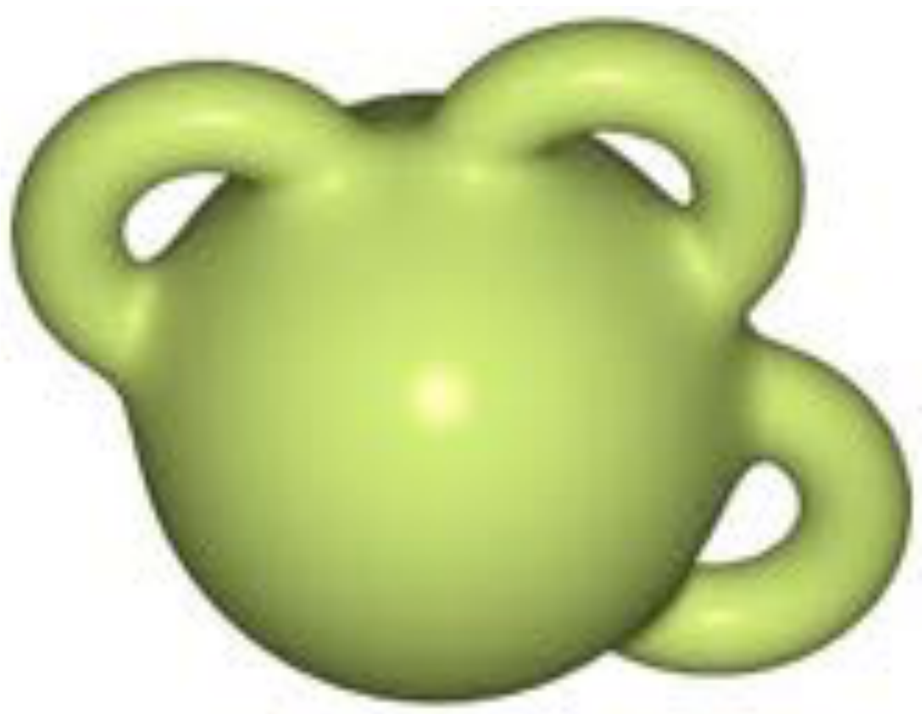
Non-orientable manifolds.....

Cannot be built from pants and caps,
instead,
use pants, caps and cross-caps

A simpler decomposition of orientable (nice) manifolds:

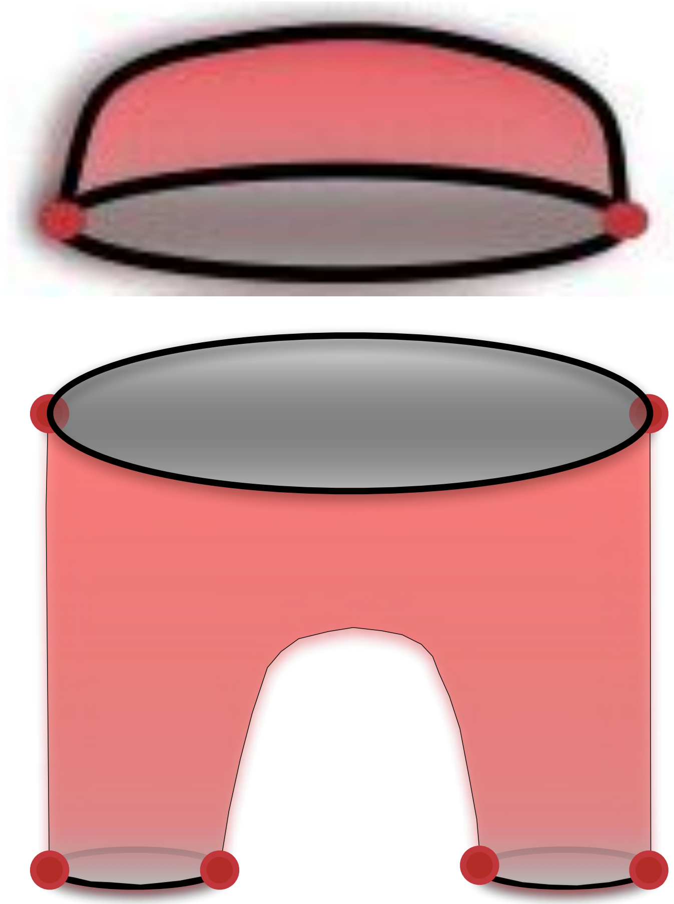
use handles and caps

ANY nice manifold= sphere + handles (+ boundary)



sphere + 3 handles (genus 3)

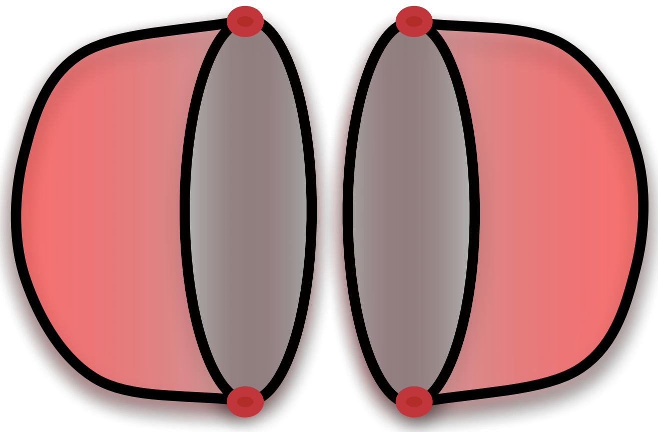
replace pants by “handle”:



handle = 1 pant + 1 cap

To build any nice manifold:

1. Start with a sphere (2 caps)



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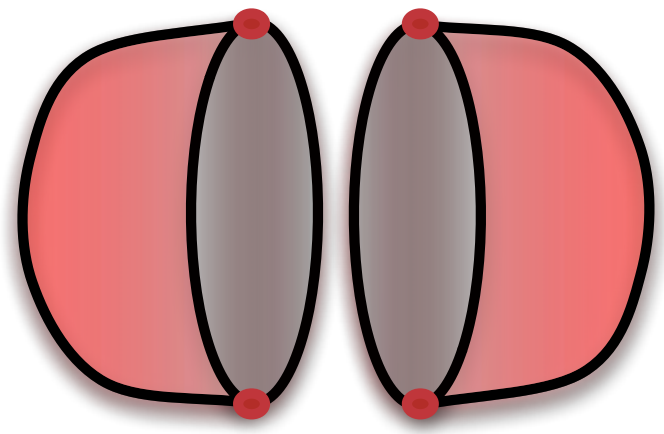
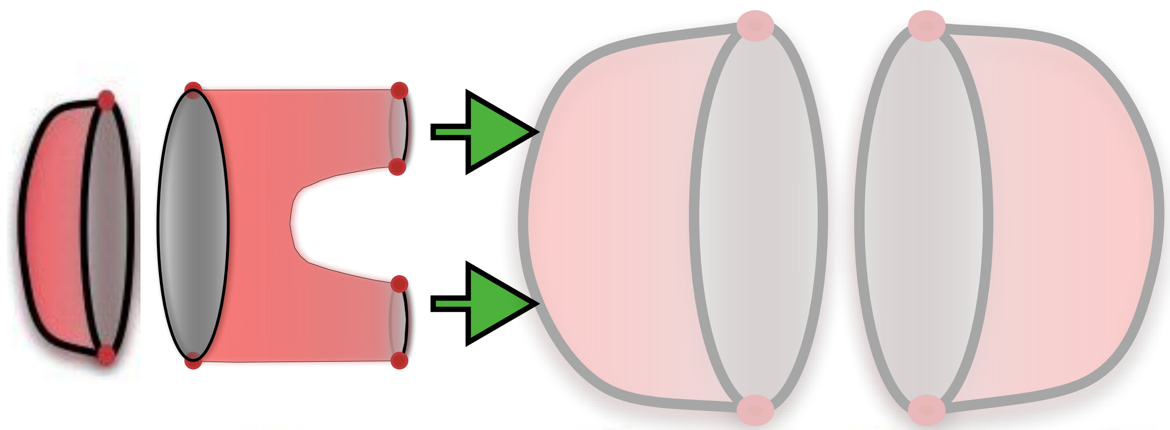


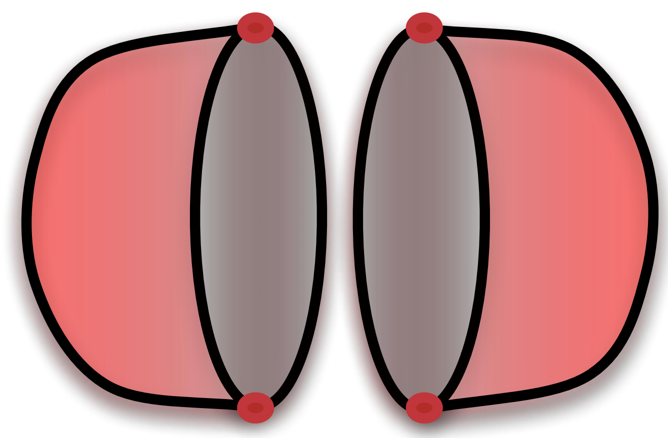
2. Add handles
(2 punctures,
1 pant + 1 cap)



$$\chi_{handle} = 2\chi_{puncture} + \chi_{pant} + \chi_{cap} = -2 - 1 + 1 = -2$$

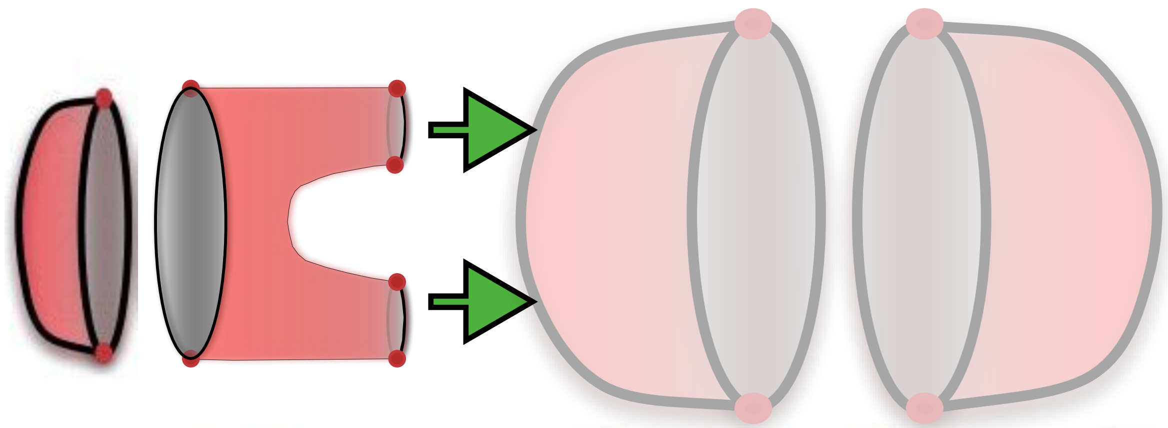
To build any nice manifold:

1. Start with a sphere (2 caps)



$\chi_{sphere} = 2$

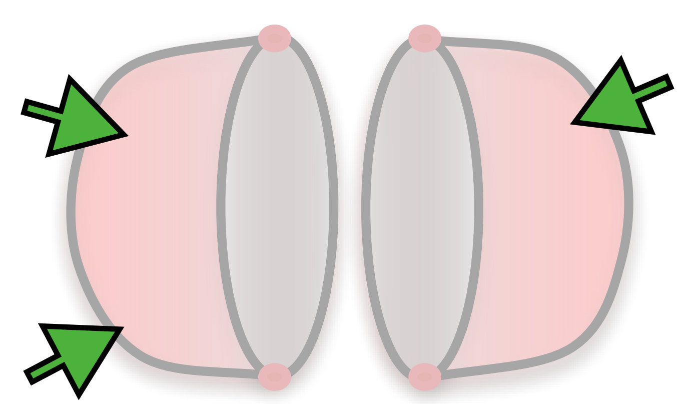
2. Add handles
(2 punctures,
1 pant + 1 cap)



$\chi_{handle} = -2$

$$\chi_{handle} = 2\chi_{puncture} + \chi_{pant} + \chi_{cap} = -2 - 1 + 1 = -2$$

3. Add boundary punctures



$\chi_{boundary} = -1$

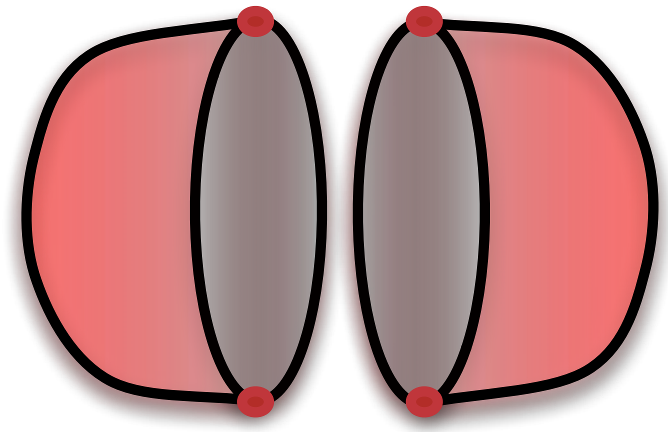
ANY nasty manifold= sphere + handles +xcaps (+ boundary)

Paul Nylander

e.g. Boys surface
= sphere + xcap

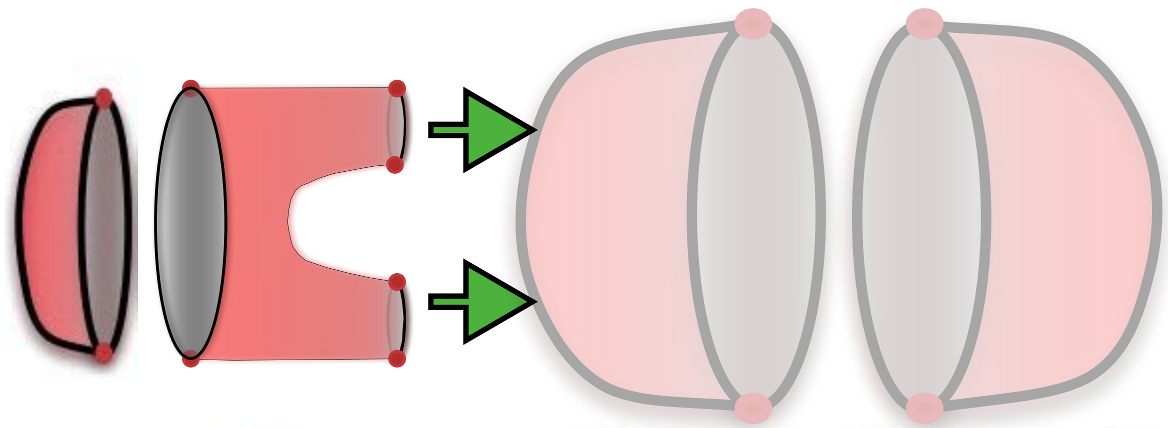
To build any manifold:

1. Start with a sphere (2 caps)



$\chi_{sphere} = 2$

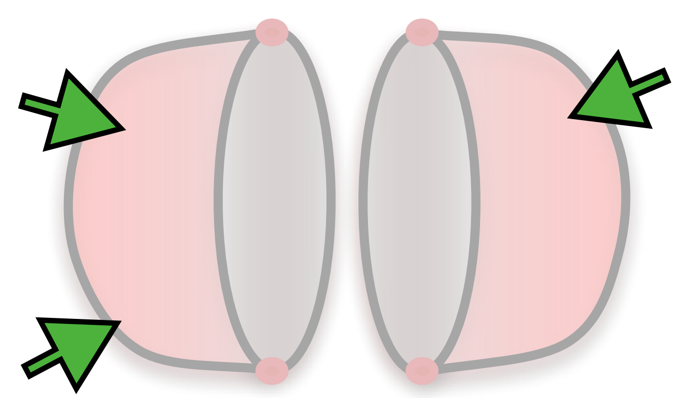
2. Add handles
(2 punctures,
1 pant + 1 cap)



$\chi_{handle} = -2$

$$\chi_{handle} = 2\chi_{puncture} + \chi_{pant} + \chi_{cap} = -2 - 1 + 1 = -2$$

4. Add boundary punctures

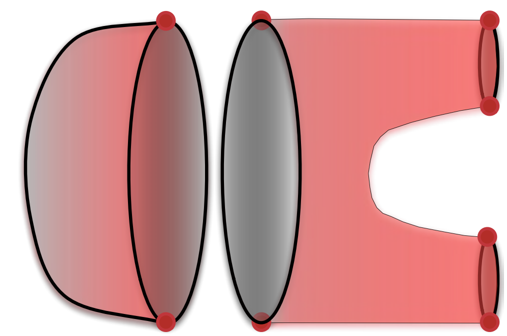
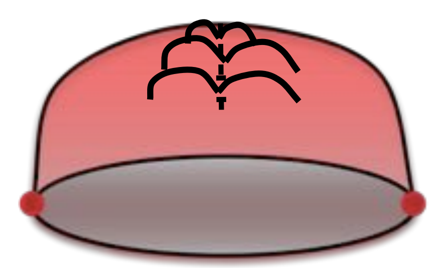
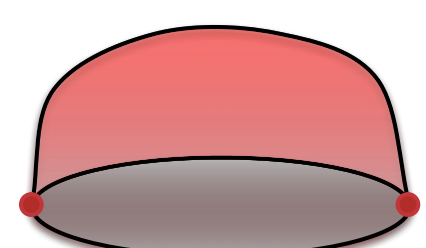


$\chi_{boundary} = -1$

ANY manifold= sphere + handles + xcaps + boundary!

$$\boxed{\chi = 2 - 2(\#handles) - (\#xcaps) - (\#boundaries)}$$

Conway symbols

modules	symbol	
handle	o	
cross-capa	x	
boundary	*	remove a 

$$\boxed{\chi = 2 - 2(\#handles) - (\#xcaps) - (\#boundaries)}$$

Notice that manifold features (handles, boundaries, xcaps) induce negative χ

Conway symbol describes the manifold

o = handle

* = boundary loop

x= crosscap

e.g. o***xx

$$\chi(\circ * * * \times \times) = 2 - [2(1) + 1(3) + 1(2)] = -5$$

better ***xxxx

Summarising:

- Manifold *topology* is independent of shape details
- Quantify topology by
  - *nice*: pants, caps and holes only
    ★2-sided, orientable
  - *nasty*: pants, caps, holes & crosscaps
    ★1-sided, non-orientable
- Characterise topology by value of χ
- Characterise *geometry* by *sign* of χ

Topology (χ)
is related to
Gaussian curvature (κ)

Gauss-Bonnet Theorem:

valid for any compact manifold

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M)$$

Surface integral of Gaussian curvature

Line integral of geodesic curve around boundary

Euler characteristic

::Gauss-Bonnet Theorem for a boundary-free manifold::

"average" geometry from topology

$$\int\int K da = 2\pi\chi$$

Euler characteristic

Surface integral of Gaussian curvature

$\int\int K da$ = solid angle traced out by normals to surface

Figure 1.18. *Planar vs. solid angle construction. A planar angle θ is equal to the perimeter of a circular arc of radius one swept out by a radial edge. The solid angle is the area of the region on the unit sphere traced out by a radial edge that sweeps through the entire solid angle. The vertex angles of the resulting spherical polygon (in this case, a triangle) are equal to the dihedral angles between adjoining faces, β_i.*

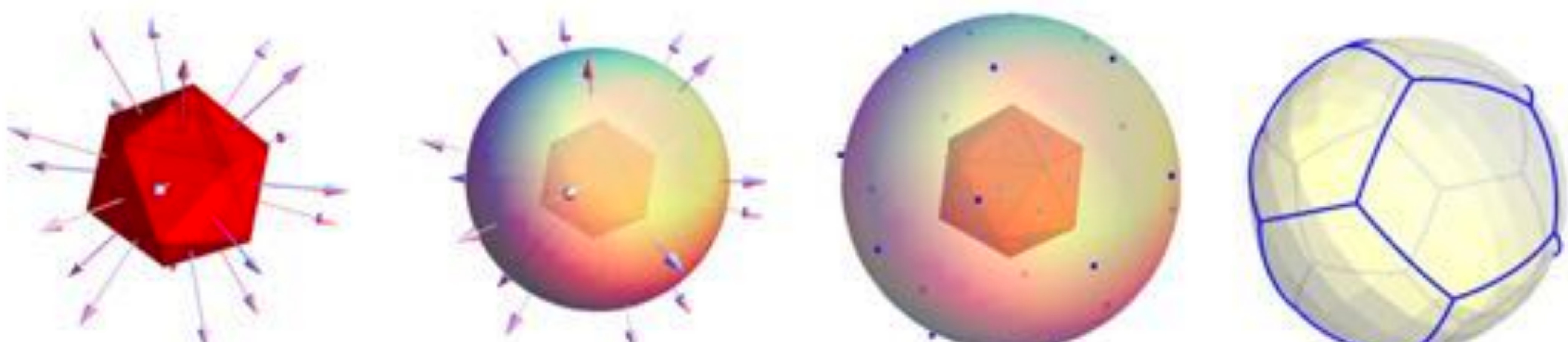
(include sign of solid angle: "Gauss map")

Area of pole region = integral (Gauss) curvature

Figure 1.22. *The pole region on the sphere due to a $\{n, z\}$ polyhedral vertex. The region is a face of a spherical polyhedron (cf. Figure 1.20, whose vertices are the pole figures of all the (z) faces that contain that vertex.*

Area of pole region of complete polyhedron = 4π

$(\ \chi = 2\ )$

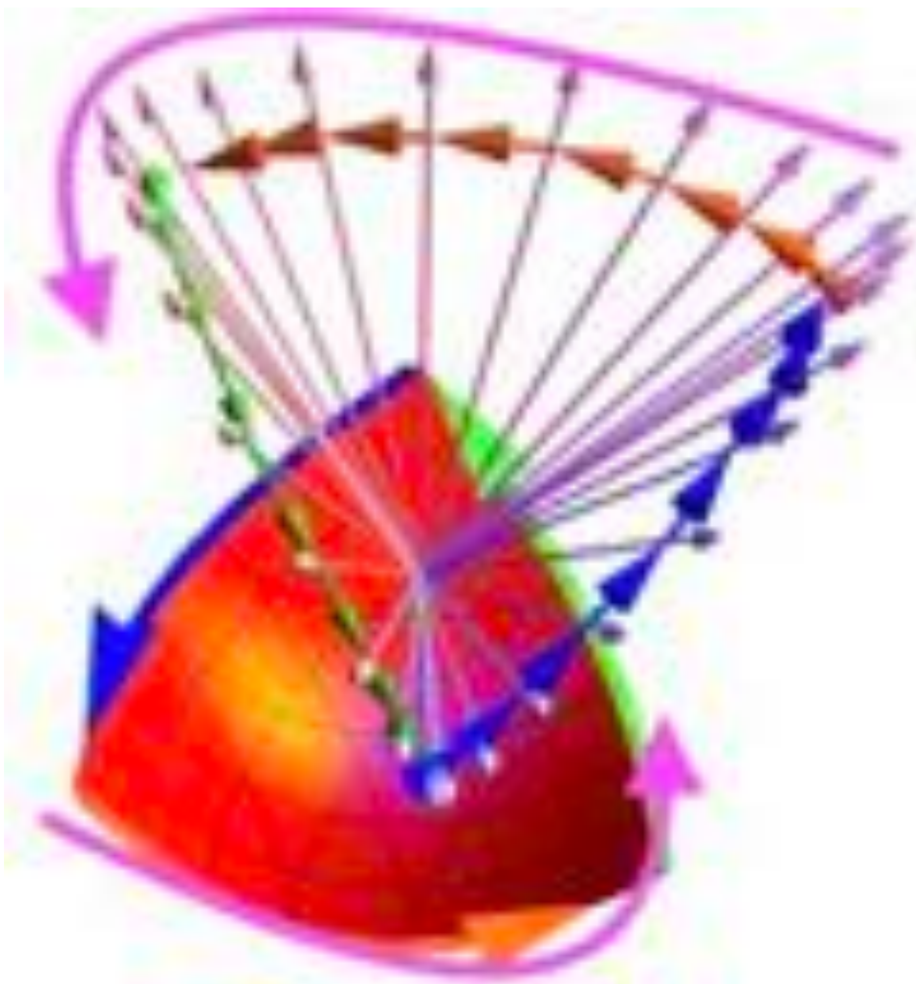


$$\int\int K da = 2\pi\chi \quad \Rightarrow \quad \boxed{< K > = \frac{2\pi\chi}{Area}}$$

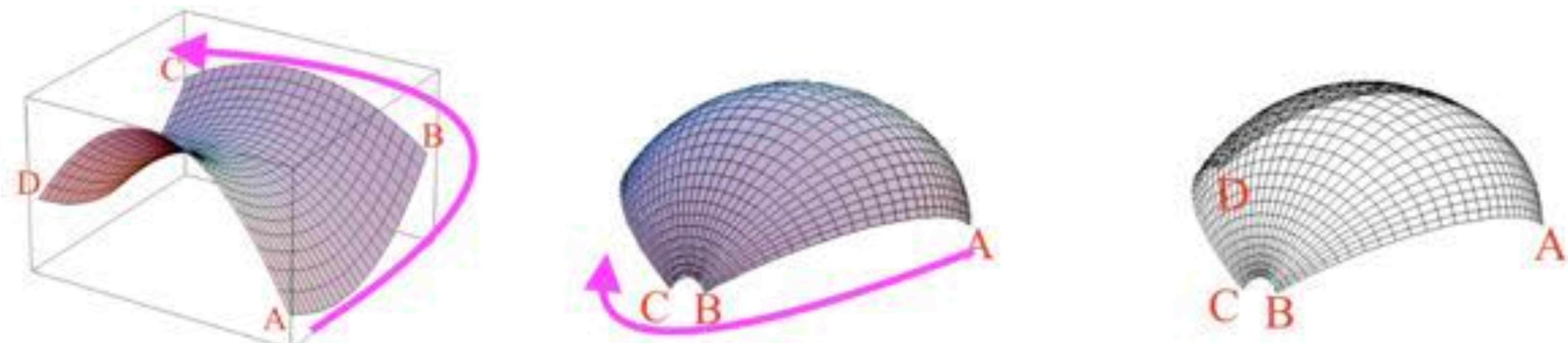
*Assume **intrinsic homogeneity: Constant K***
(i.e. no curvature variations)

$$\int\int K da = K.Area$$

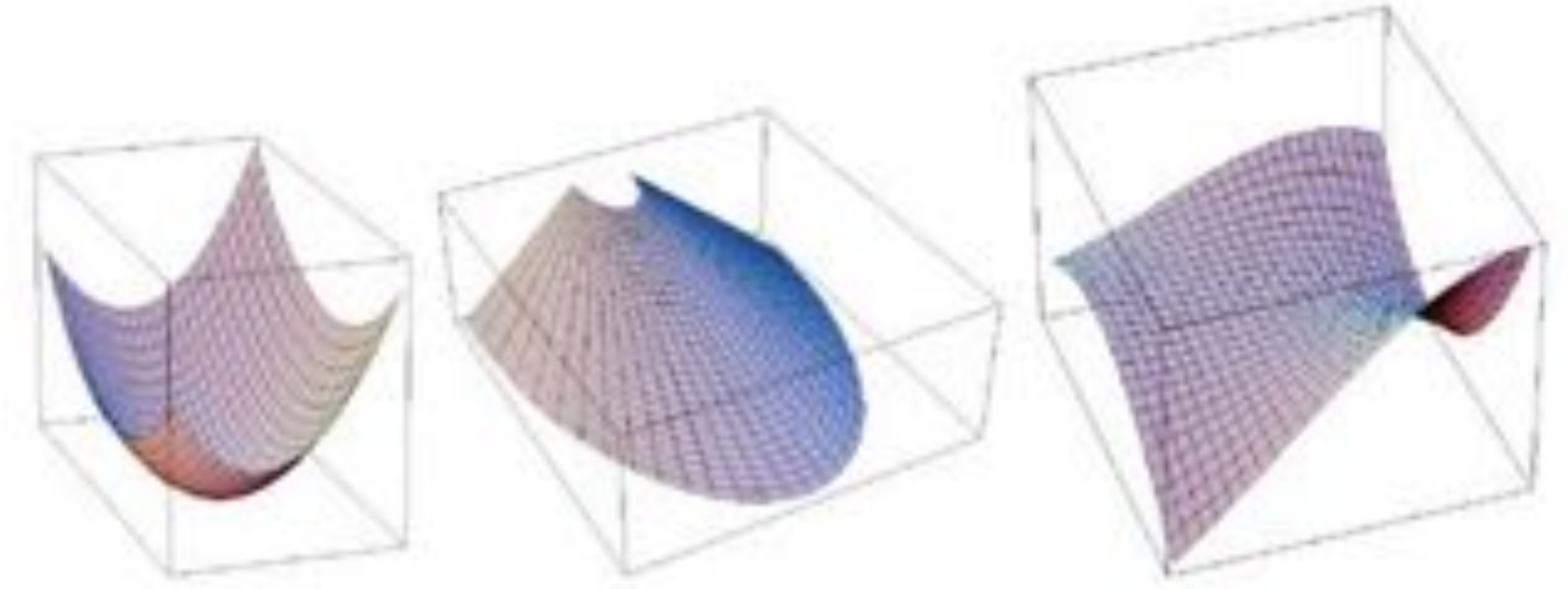
sign of solid angle:



Positive Gaussian curvature

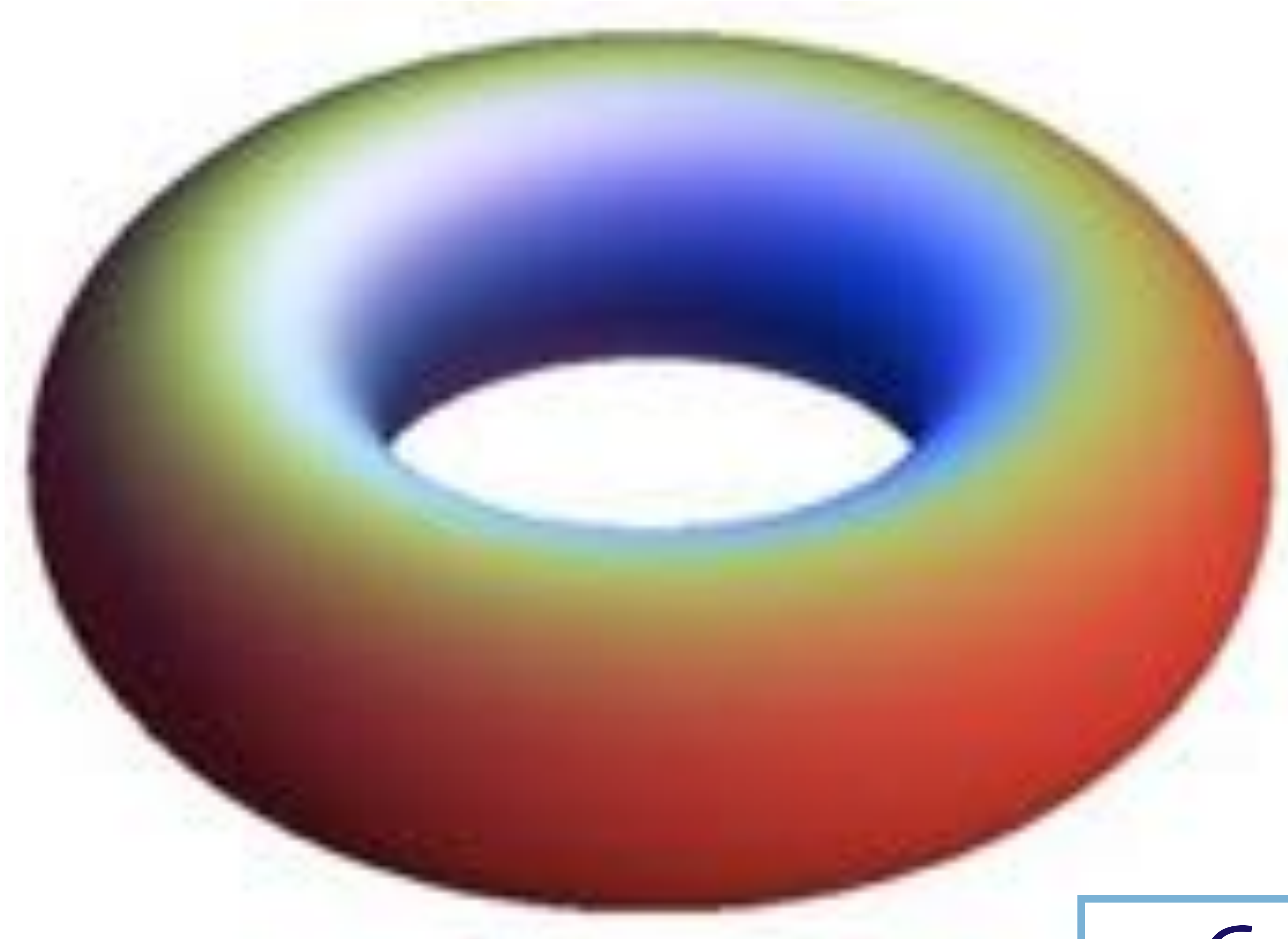


Negative Gaussian curvature

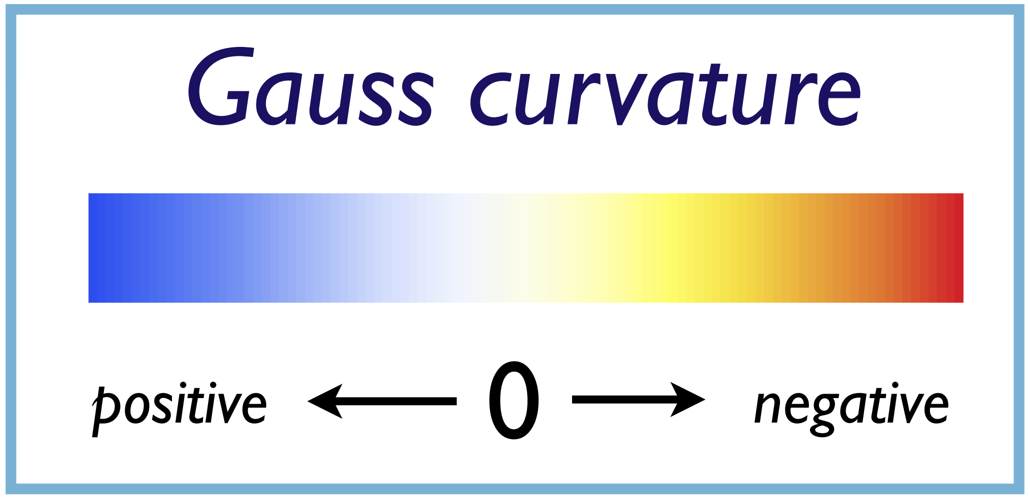


$< K > \; > 0$	$< K > \; = 0$	$< K > \; < 0$
elliptic	euclidean	hyperbolic

... a torus has $\chi = 0$ ie. it is - on average - flat!



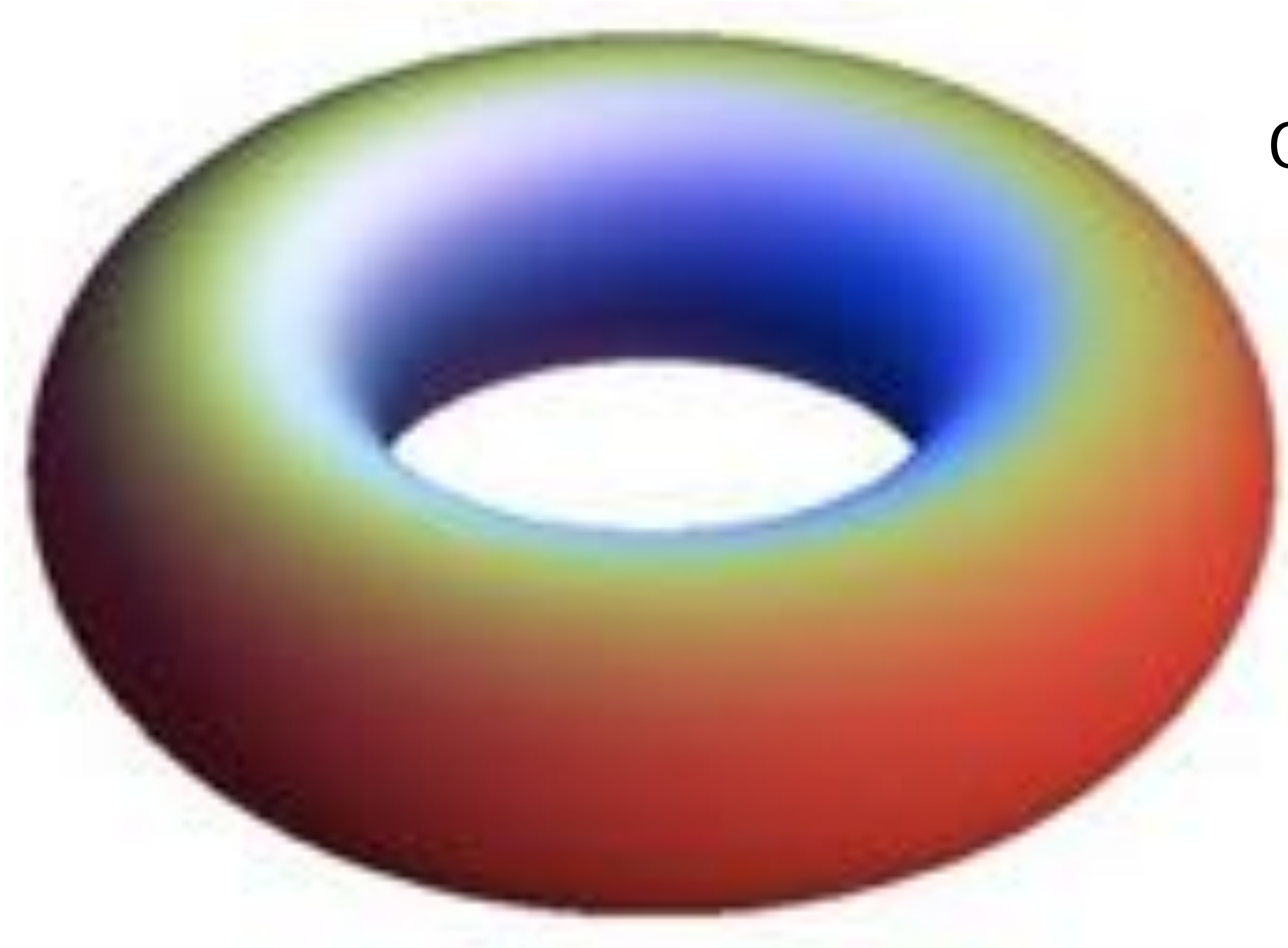
exactly equal contributions of + and - Gaussian curvature



... a torus has one handle

$$\boxed{\chi = 2 - 2(\#handles) - (\#xcaps) - (\#boundaries)}$$

$$\chi = 0$$



Conway symbol:
o

a tritorus has three handles.....

$$\boxed{\chi = 2 - 2(\#handles) - (\#xcaps) - (\#boundaries)}$$

$$\chi = -4$$

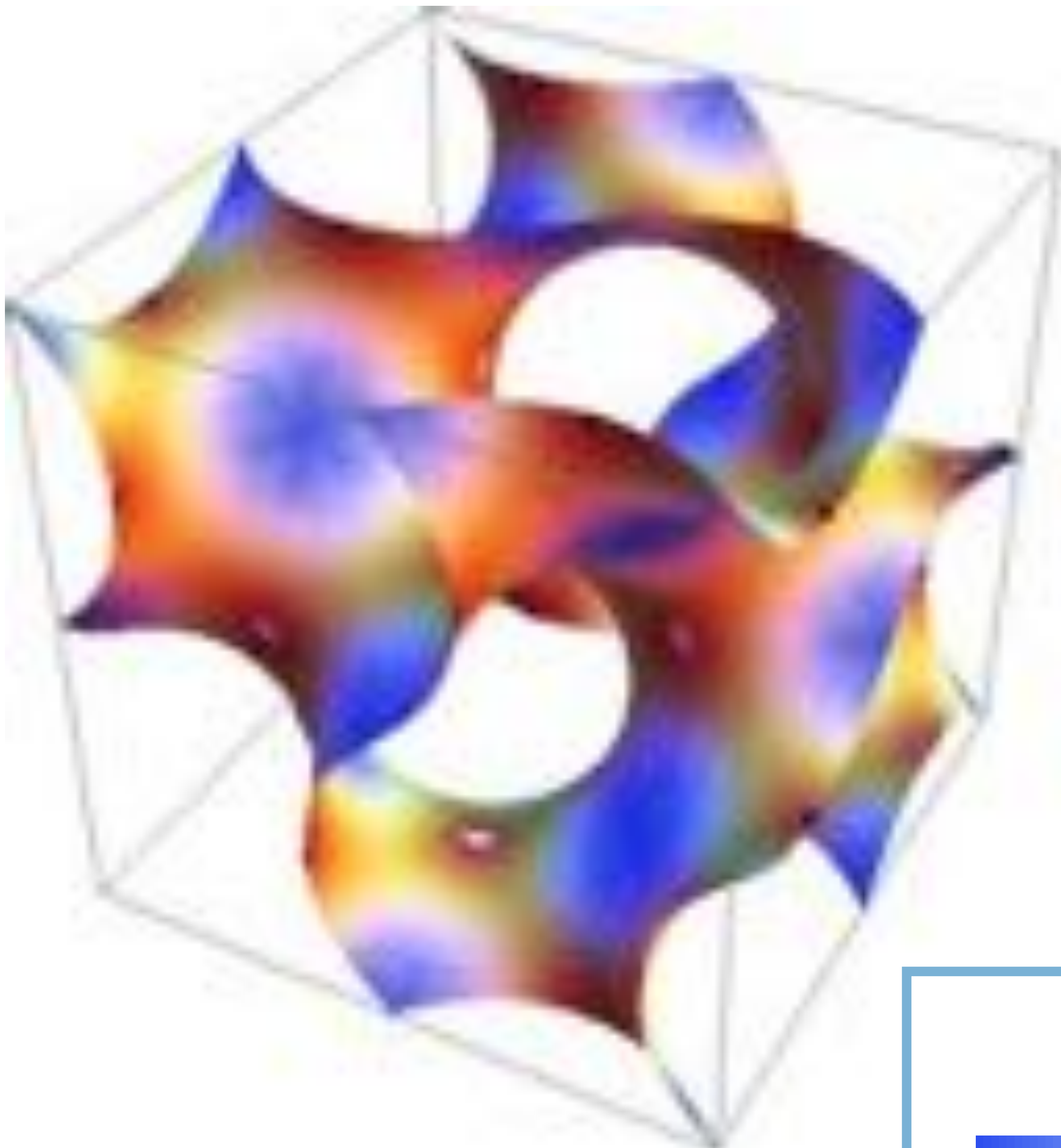
$$\boxed{< K >= \frac{2\pi\chi}{Area}}$$

... so a tritorus is hyperbolic, with negative <K>

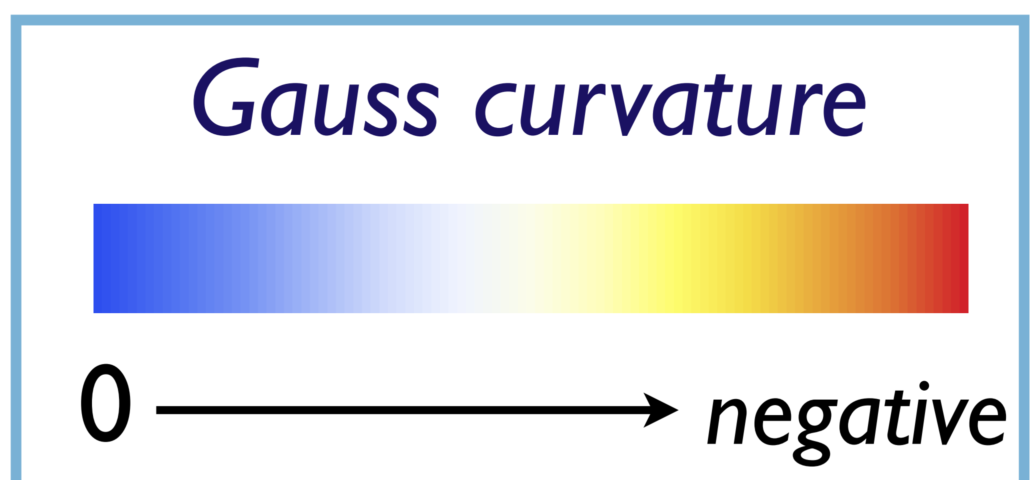
... a genus-3 tritorus is -- on average -- HYPERBOLIC

Conway symbol:
ooo

... an infinite genus 3-periodic surface is -- on average -- HYPERBOLIC



Conway symbol:
oo......



local homogeneous 2D **flat** geometry can be globally extended in 3D space:

euclidean plane = normal plane

local homogeneous 2D **elliptic** geometry can be globally extended in 3D space:

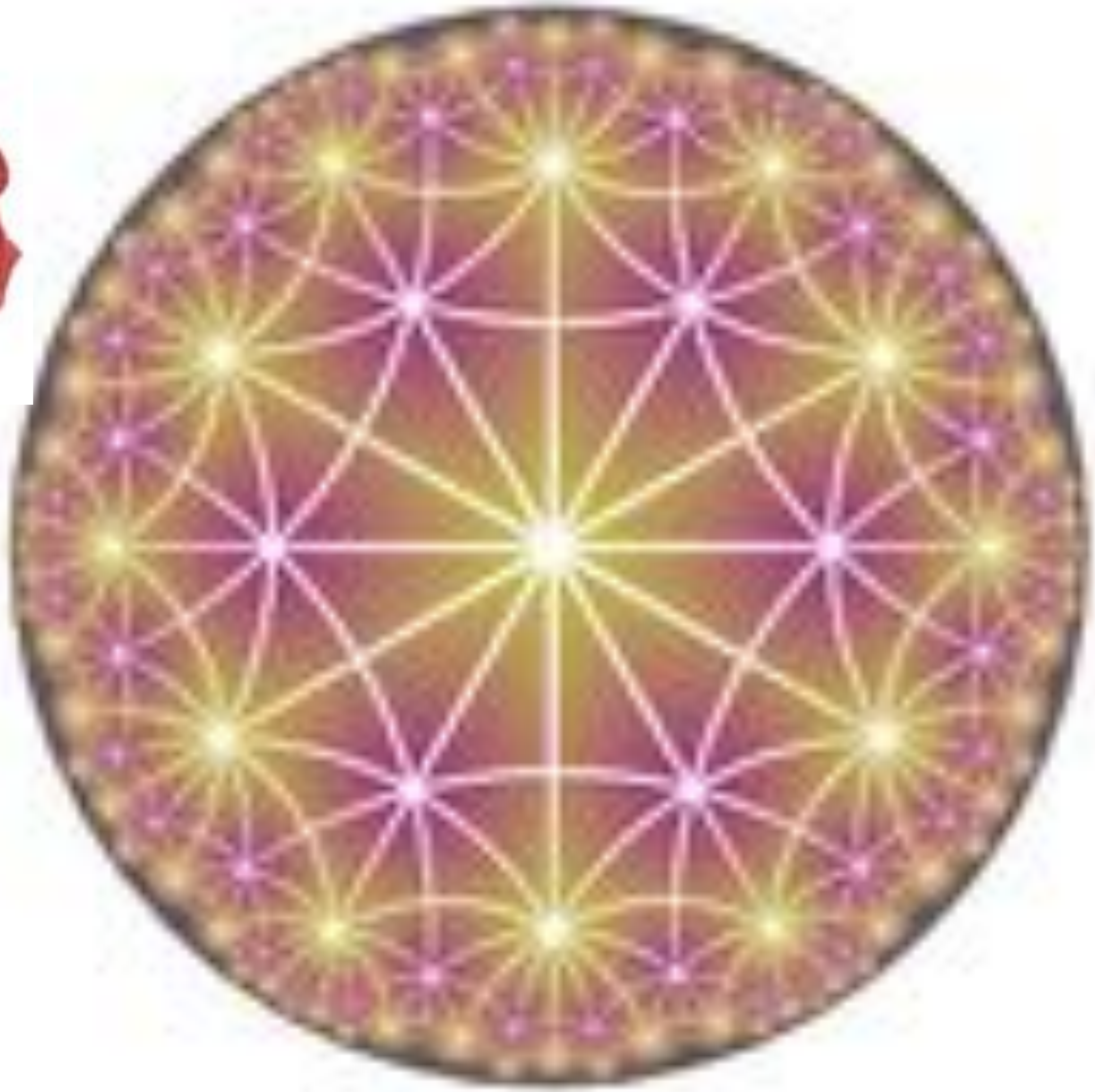
www.geom.uiuc.edu/zoo/toptype/sphere/stills.html

2D hyperbolic space is much “bigger” than 2D flat space

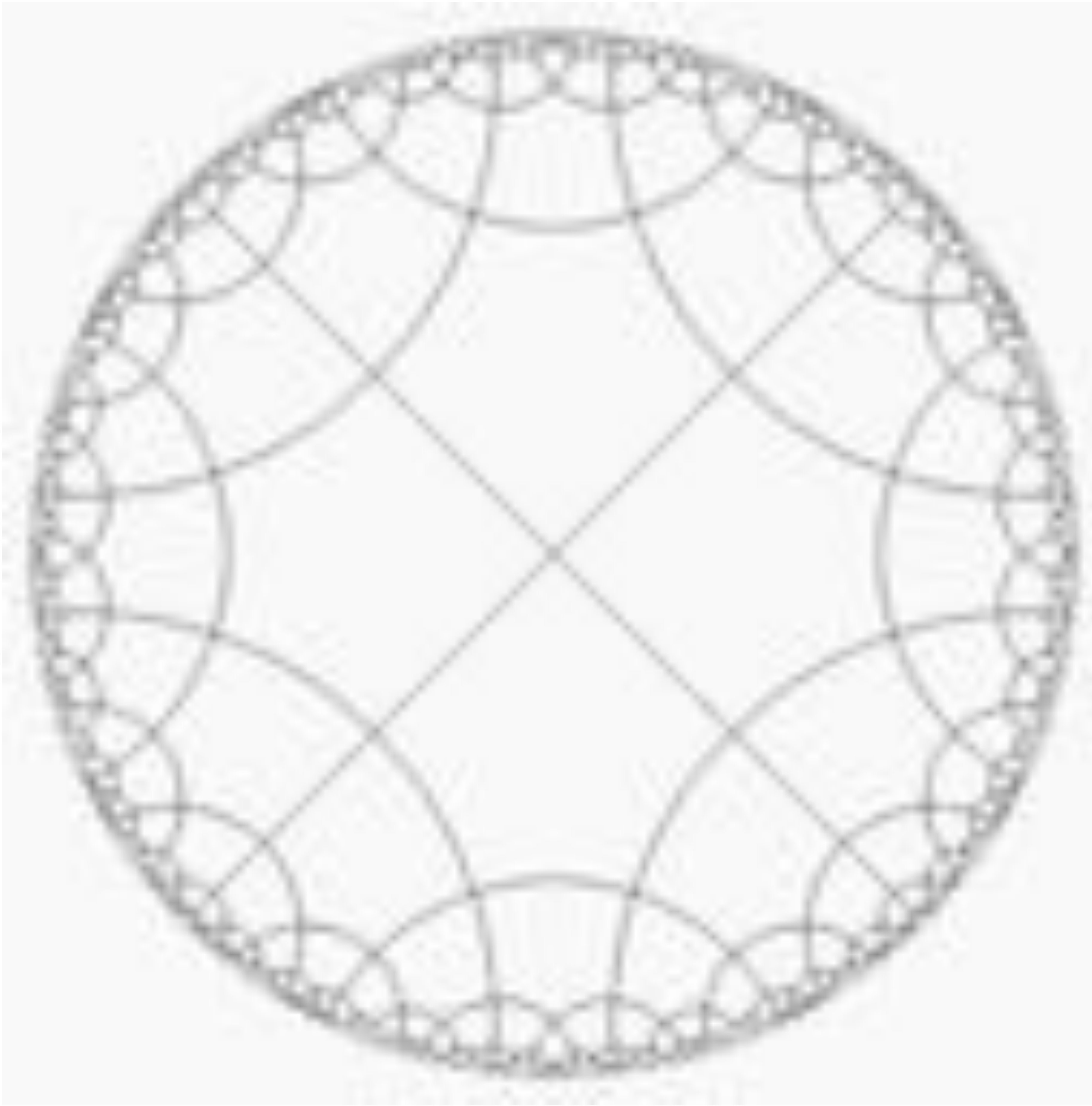
e.g. hyperbolic: area of a disc ~ exponential(radius)

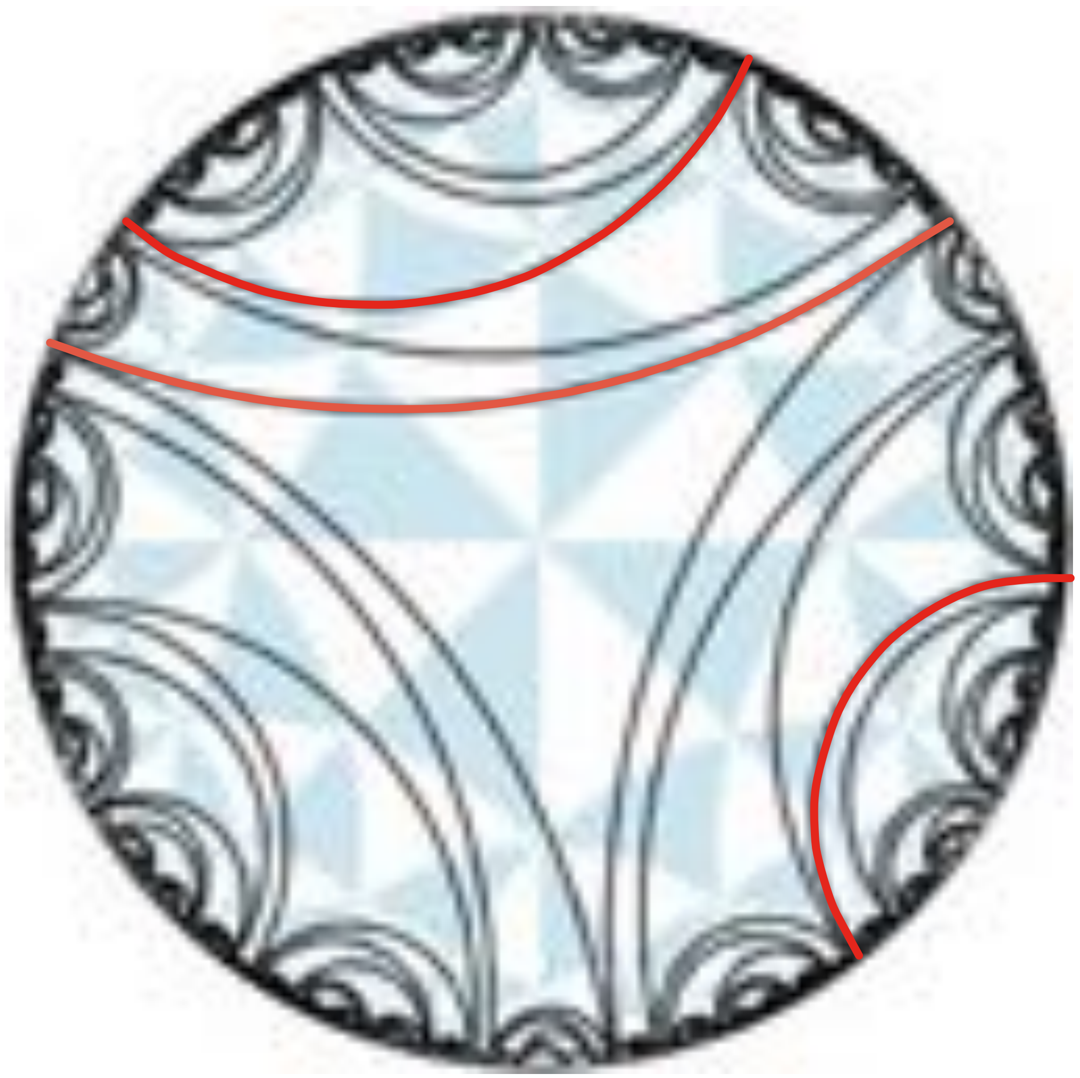
flat: area of a disc ~(radius) ^{2}

We represent the hyperbolic plane by the Poincaré model



$$ds = \frac{2|dz|}{1-|z|^2}$$





(Hyper)Parallel lines

Intersecting lines

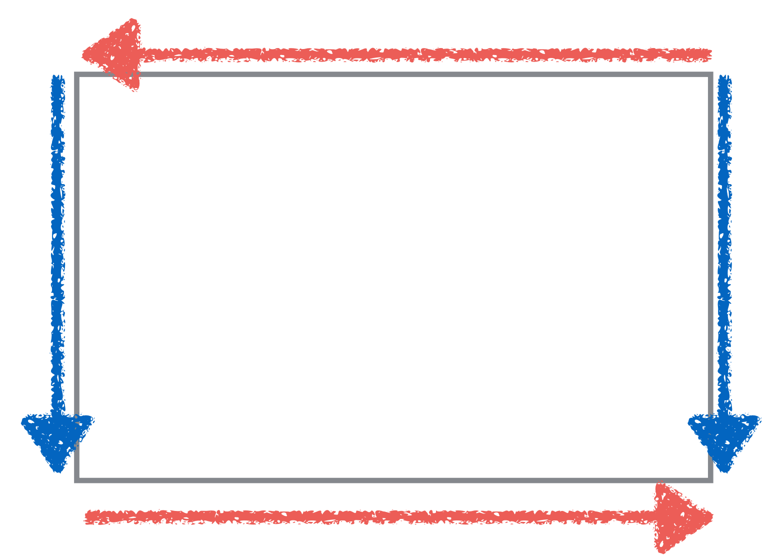
what are these manifolds?

cylinder

Moebius strip

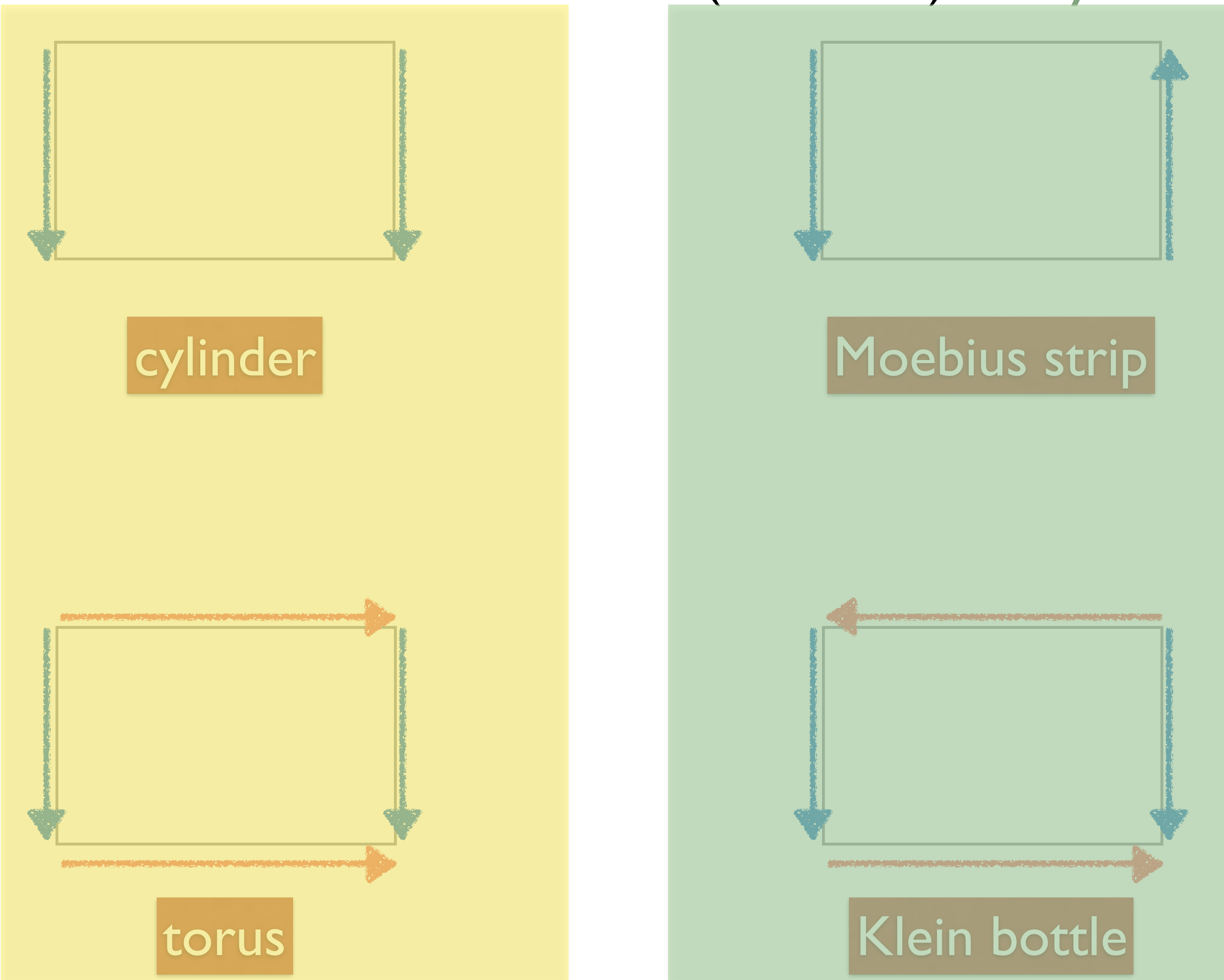
to here

torus

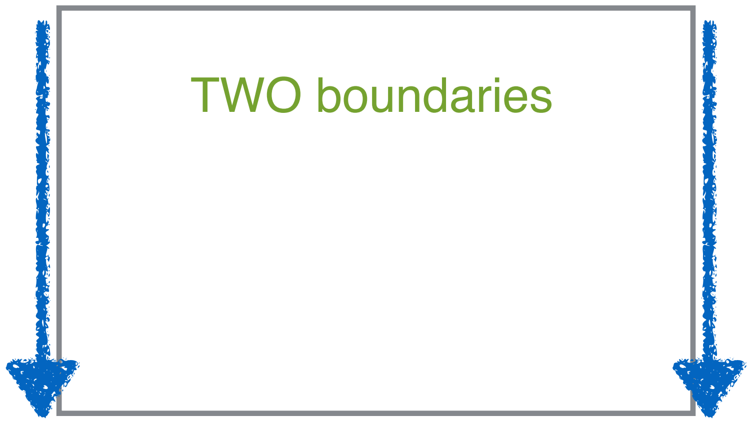


Klein bottle

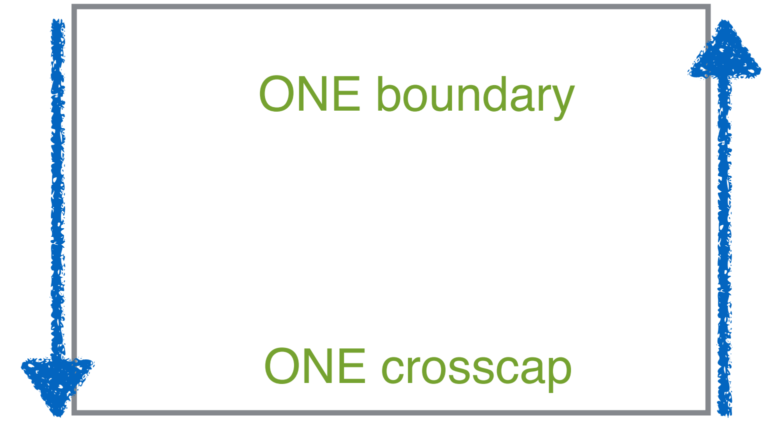
which ones are nice (oriented), nasty?



how many caps? xcaps? handles? boundaries?

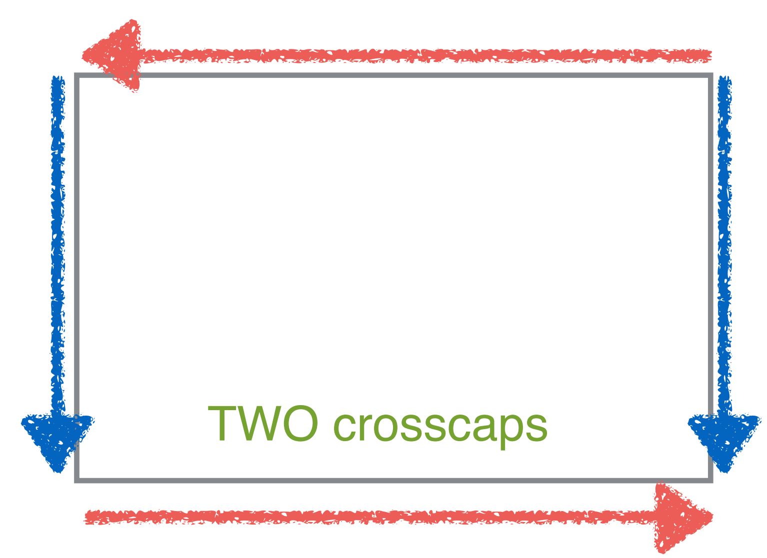


cylinder



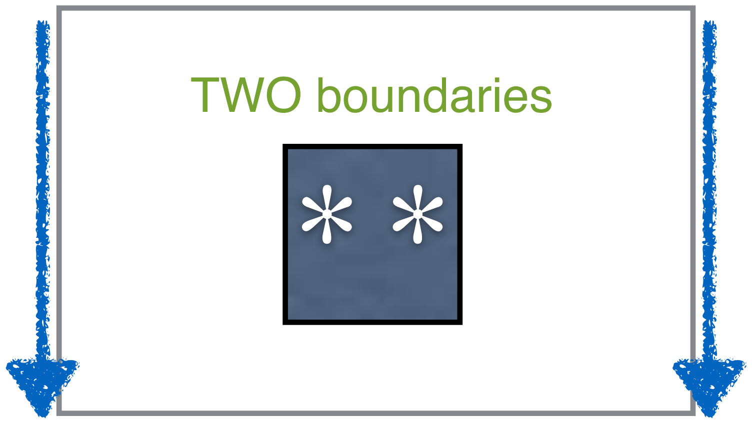
Moebius strip

torus

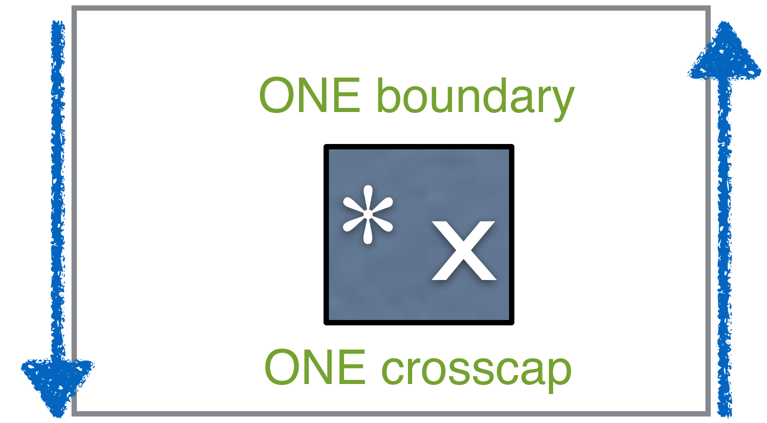


Klein bottle

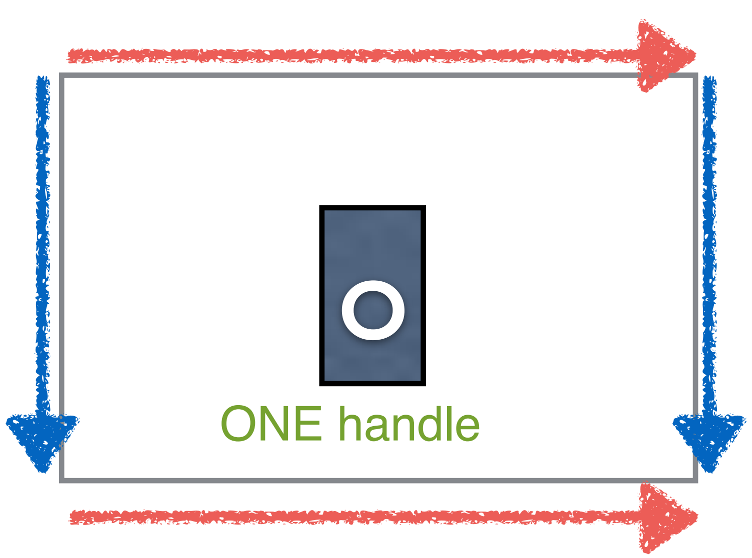
what are their Conway symbols?



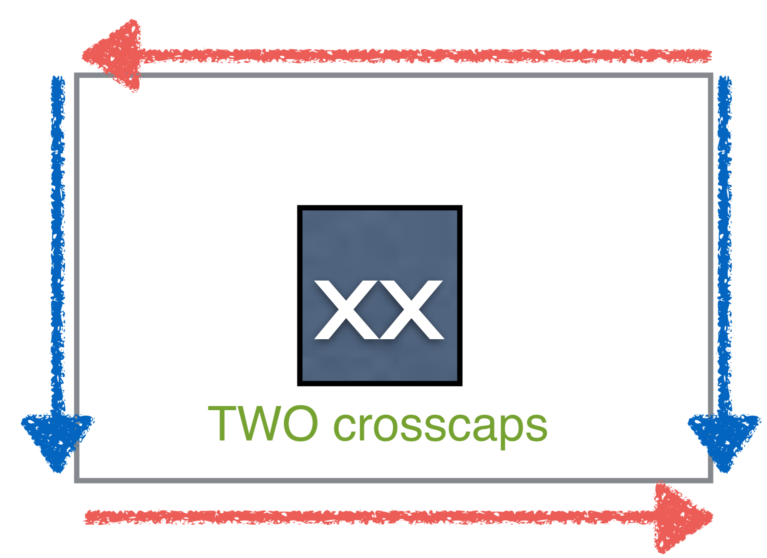
cylinder



Moebius strip

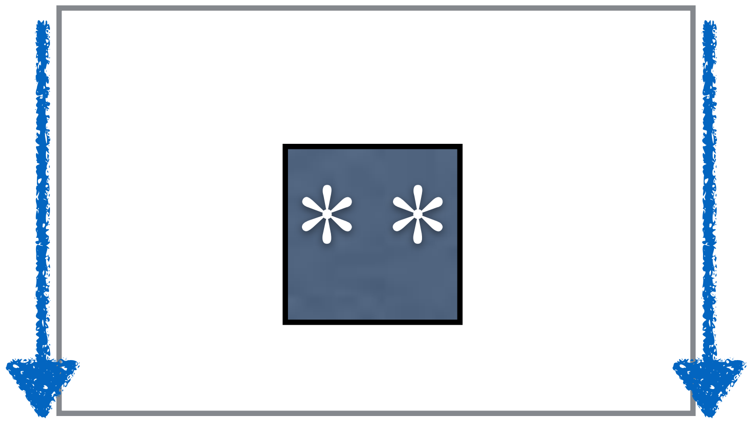


torus

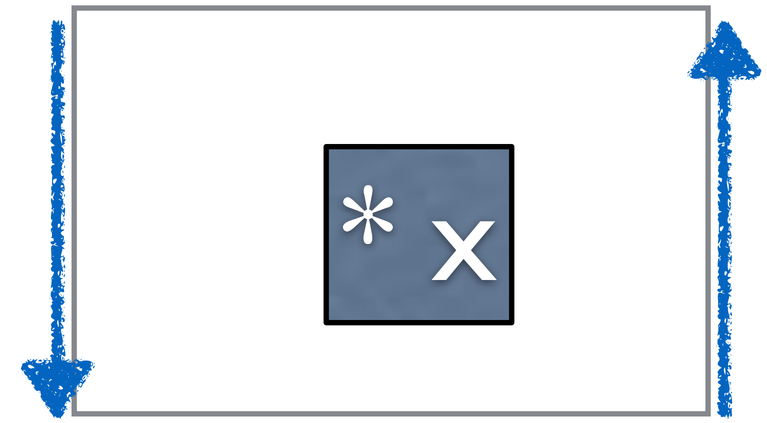


Klein bottle

what is their geometry?

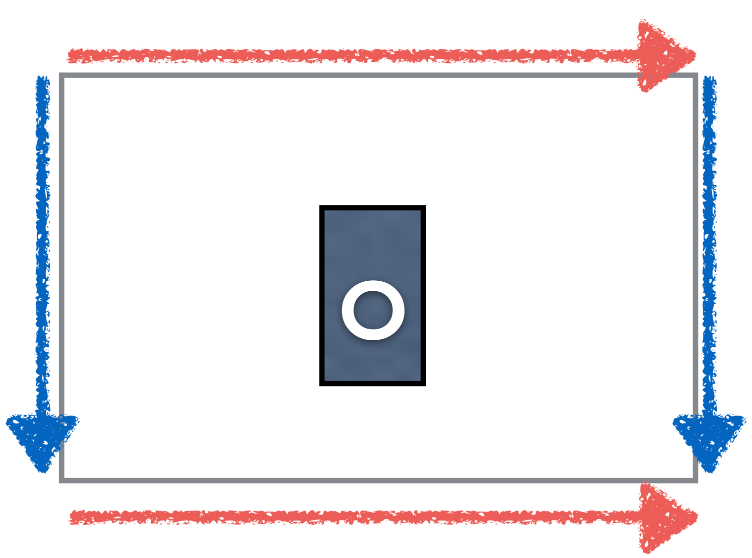


cylinder

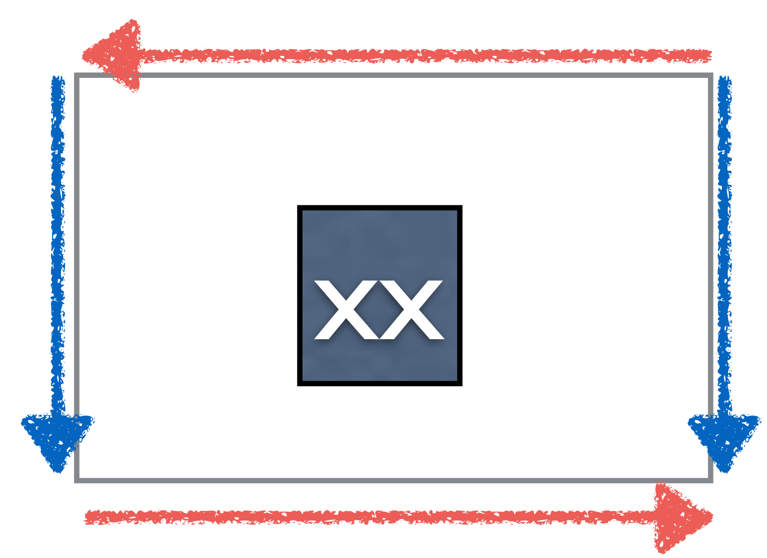


Moebius strip

euclidean, K=0

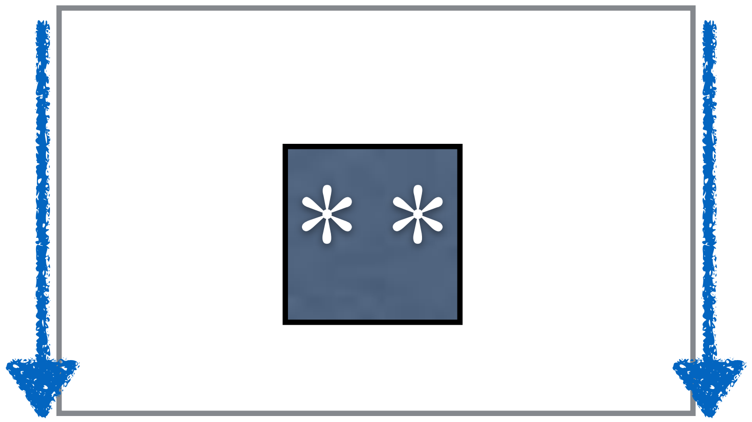


torus

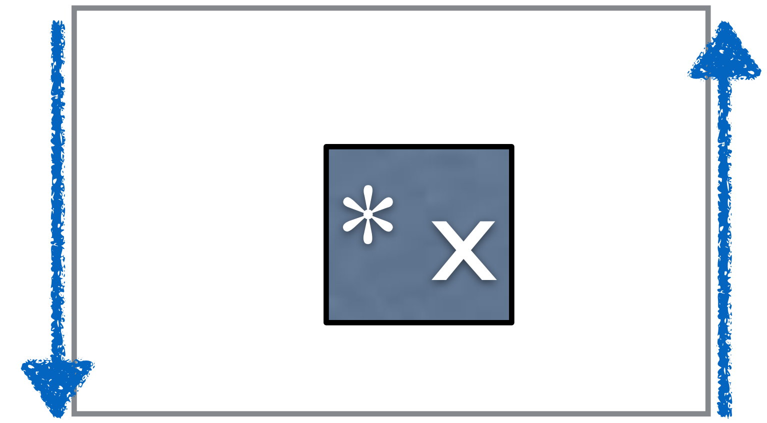


Klein bottle

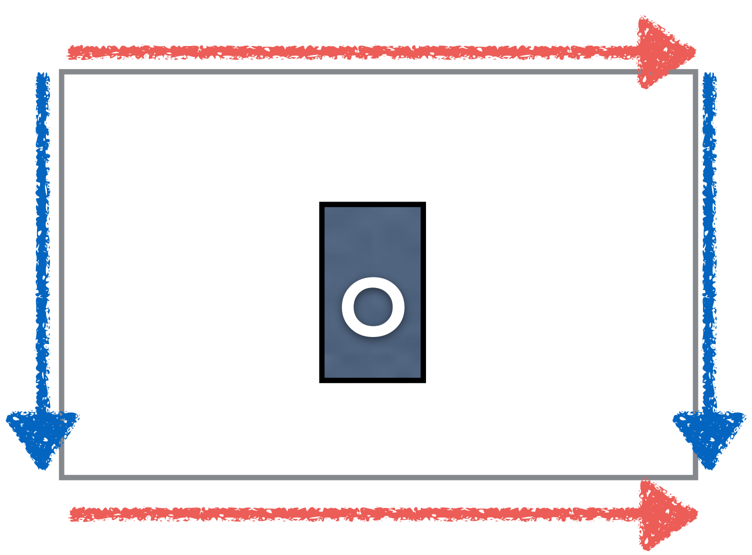
what is their geometry?



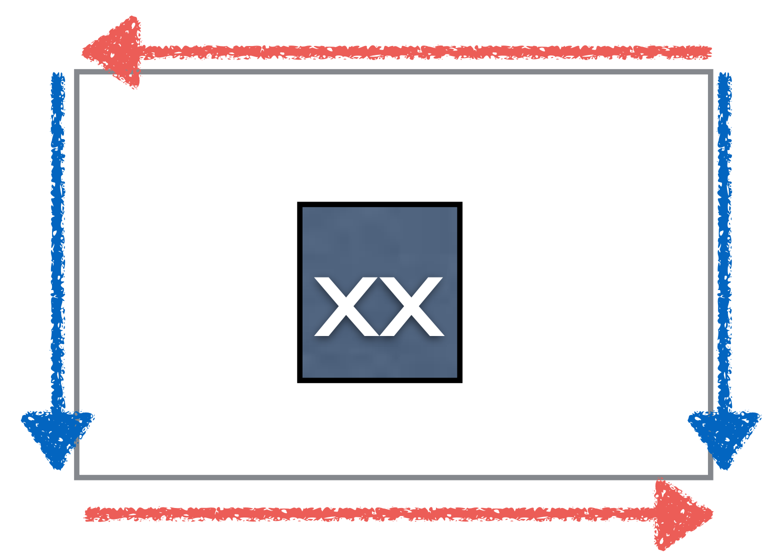
cylinder



Moebius strip

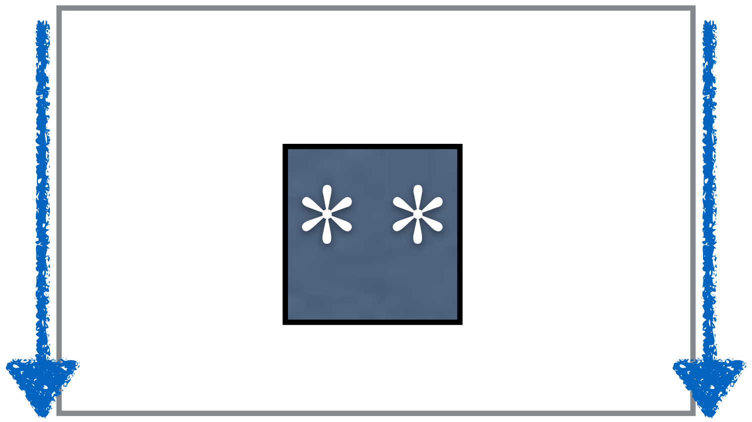


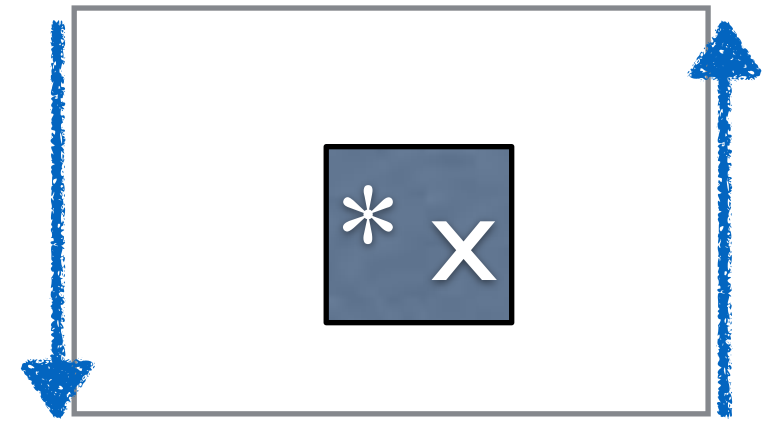
torus



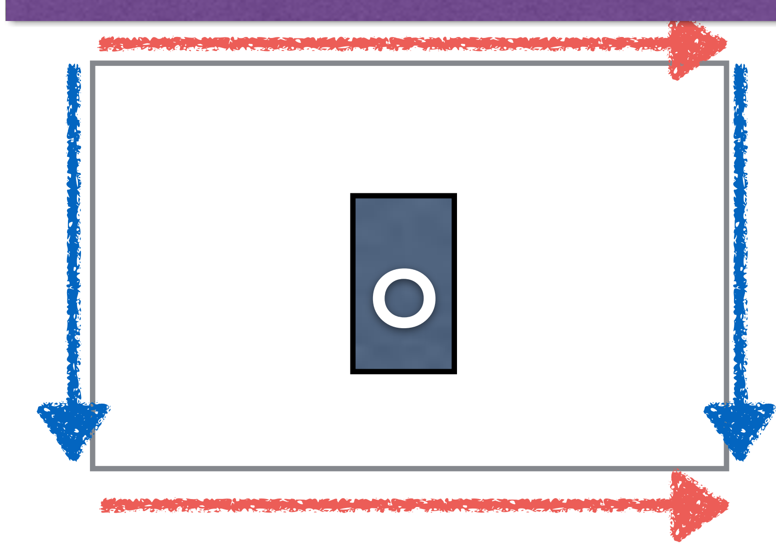
Klein bottle

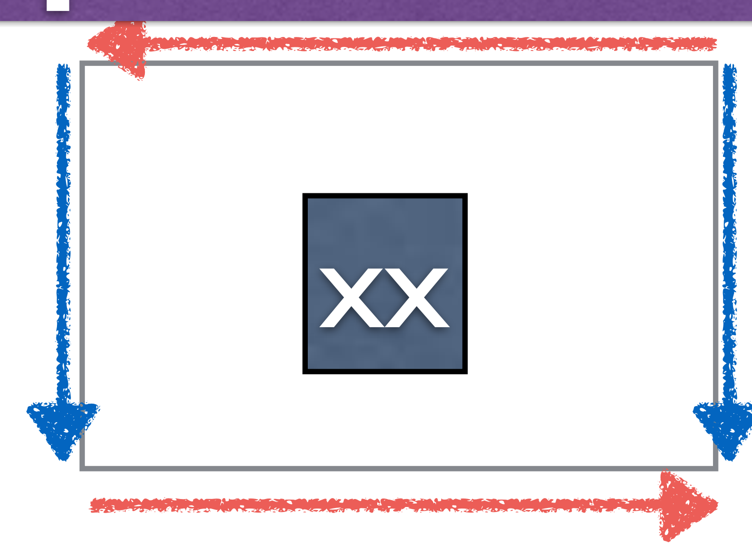
Let’s build the universal cover of these manifolds…





The universal covers tile the euclidean plane, E^2





torus

Klein bottle

covers and *universal cover* of manifolds

Exercise 1.4.2.1 The n-sheeted cyclic cover of $\mathscr{F}_2$ is defined by taking n coaxial copies of $\mathscr{F}_2$, cutting them through one of the handles (Figure 112), then identifying the boundaries of the cuts cyclically (ith on the left with $(i+1)$th on the right, nth on the left with first on the right). By computing the Euler characteristic of the cover, show that it can be an orientable surface of arbitrary genus > 1, if n is suitably chosen.

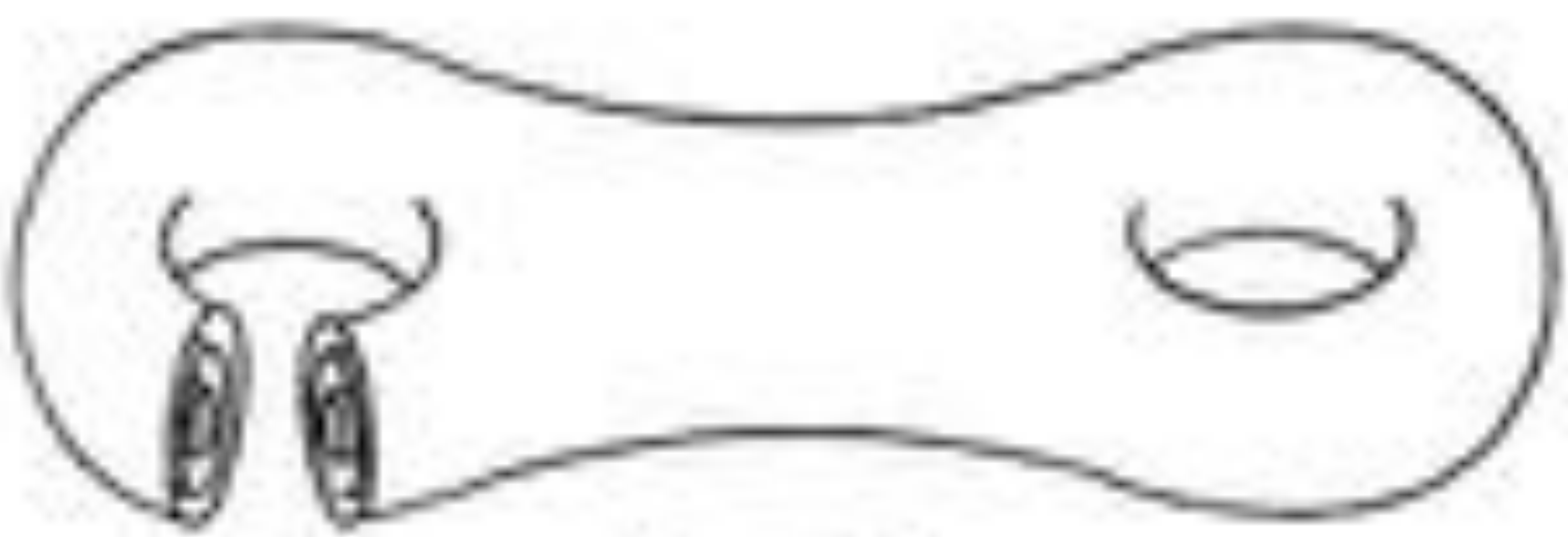
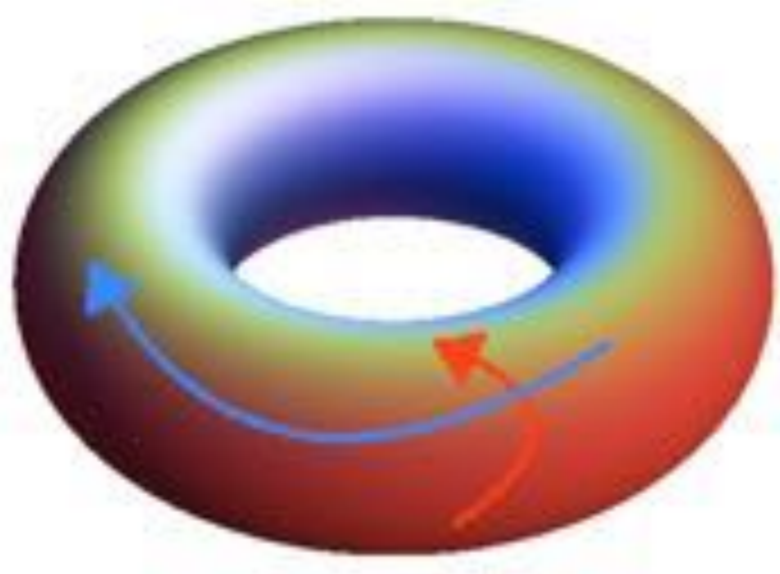


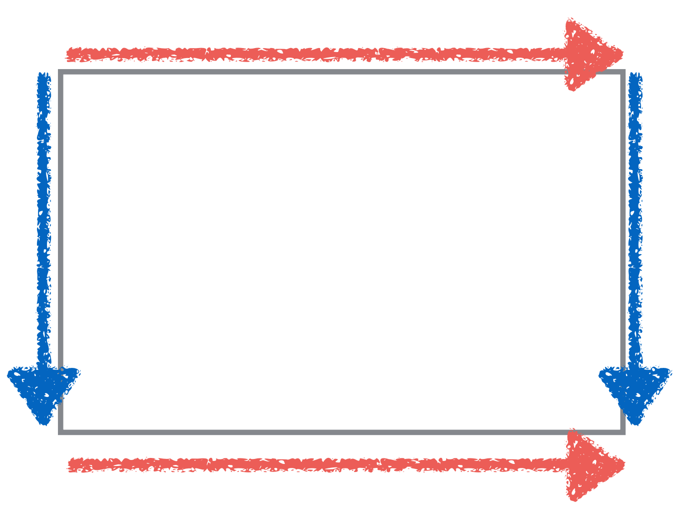
Figure 112

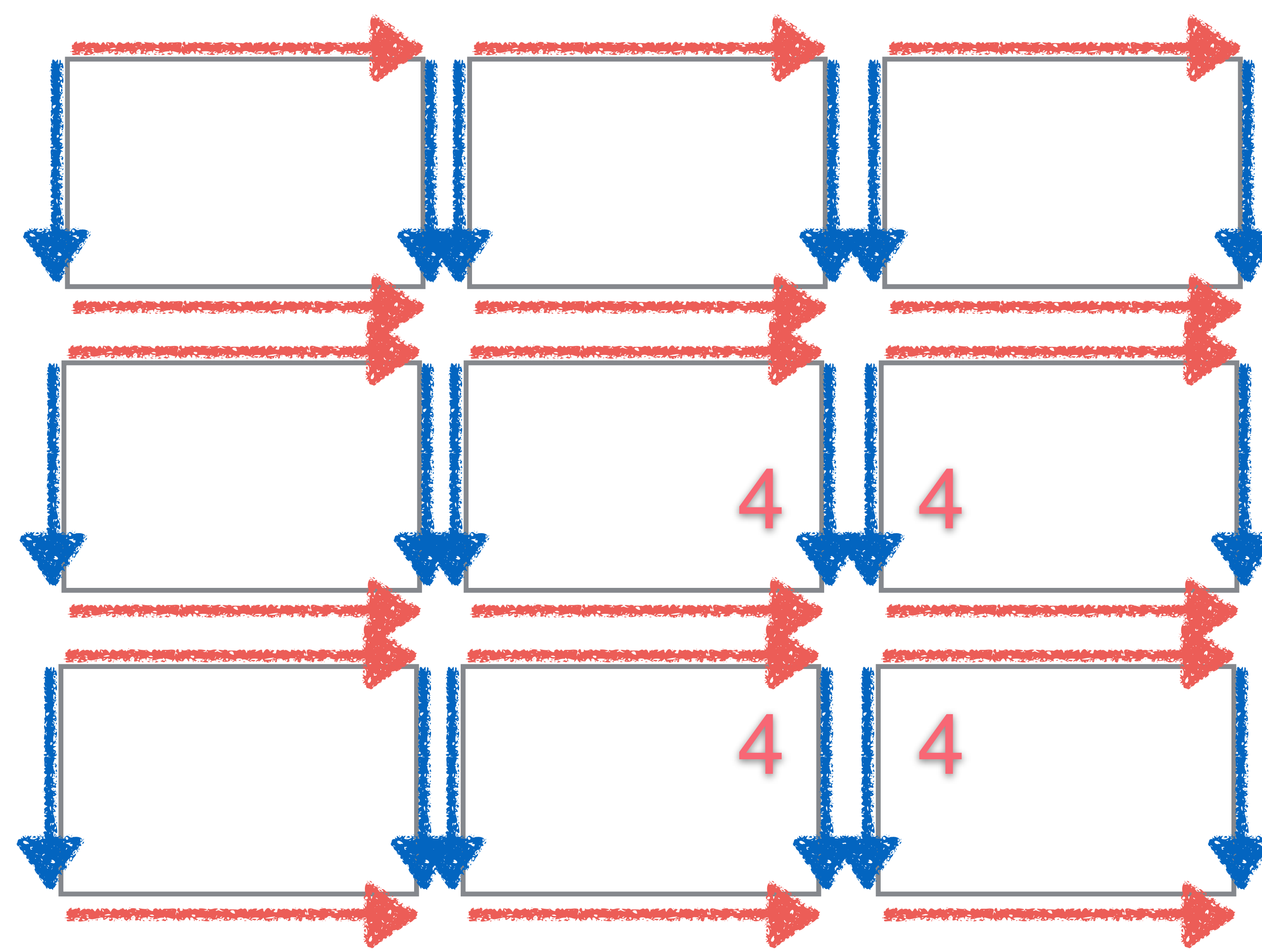
"*Classical Topology and Combinatorial Group Theory*"
John Stillwell

build the “universal cover” of o:



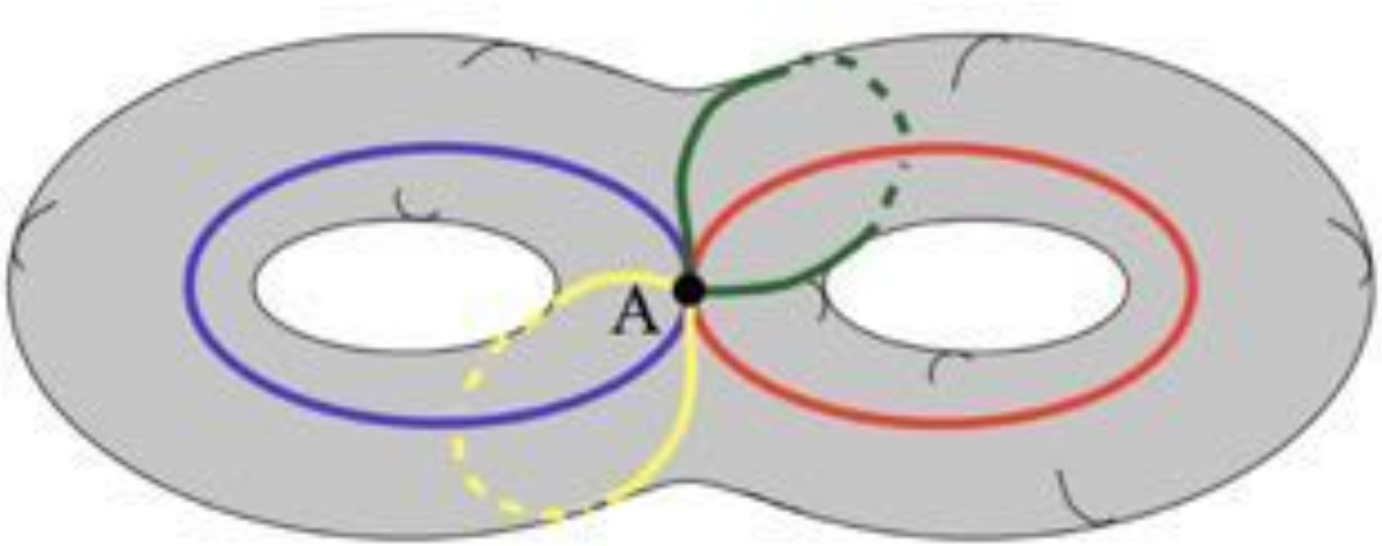
cover

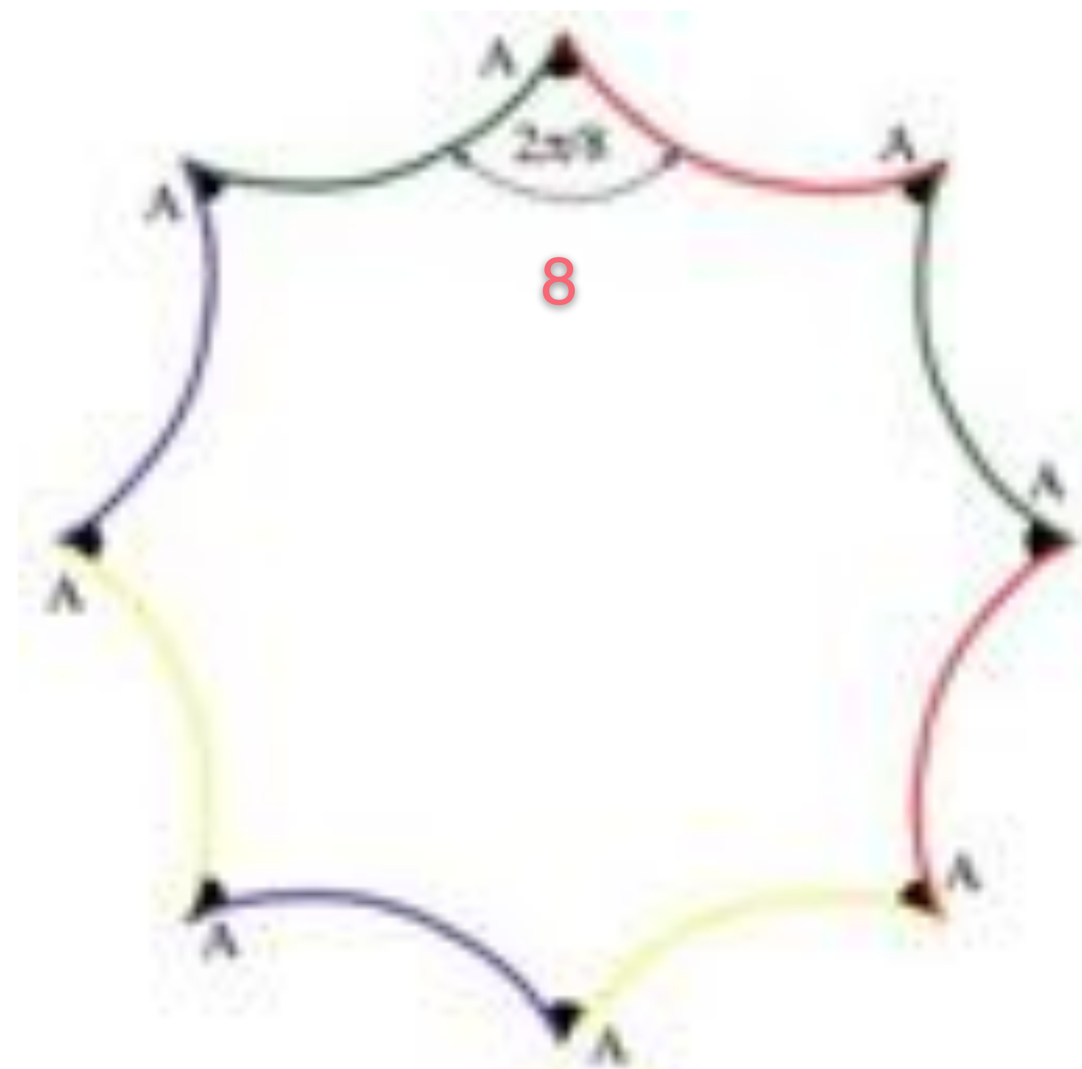




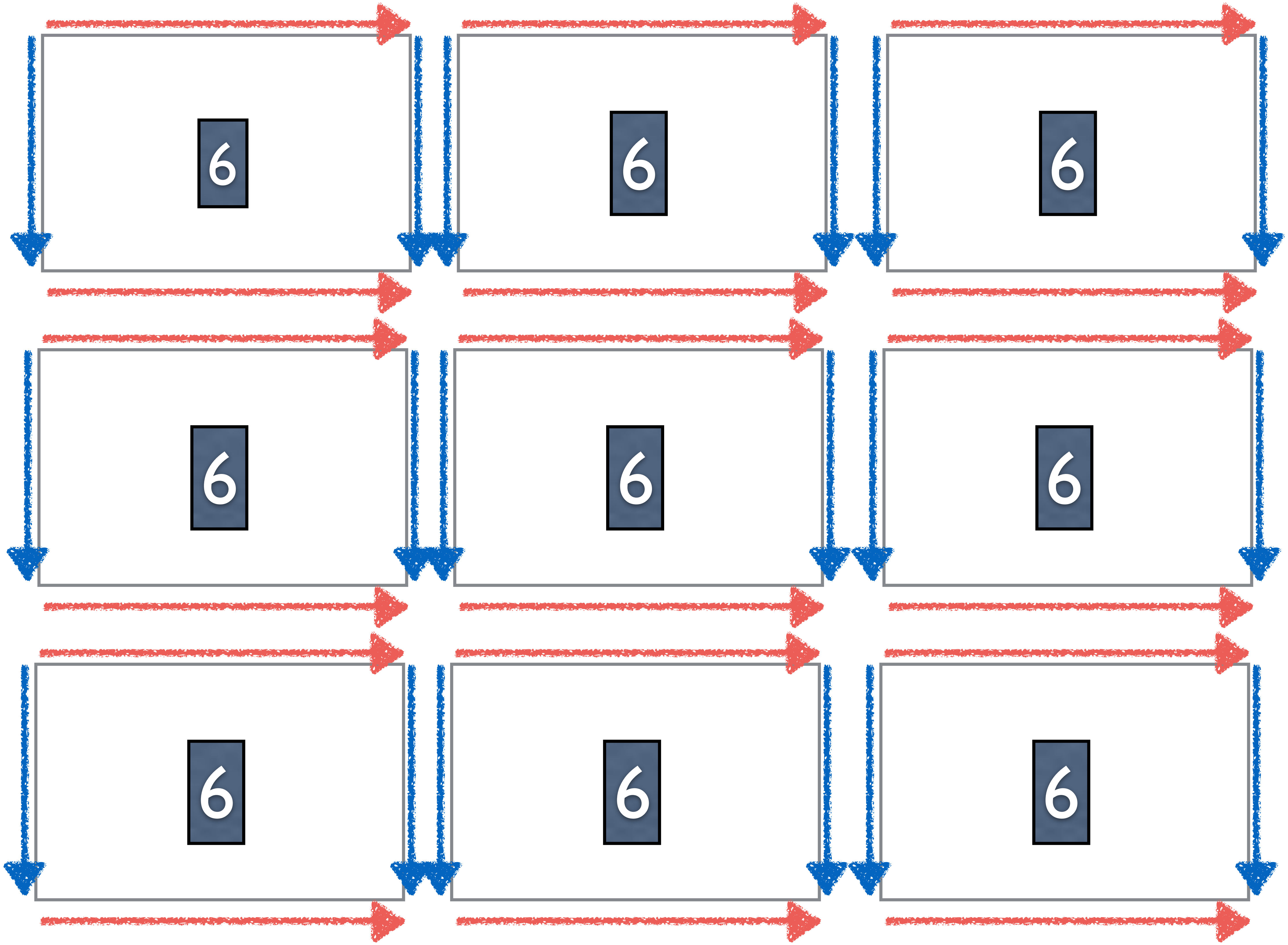
universal cover - E2 tiled by (4,4)

build the "universal cover" of oo:



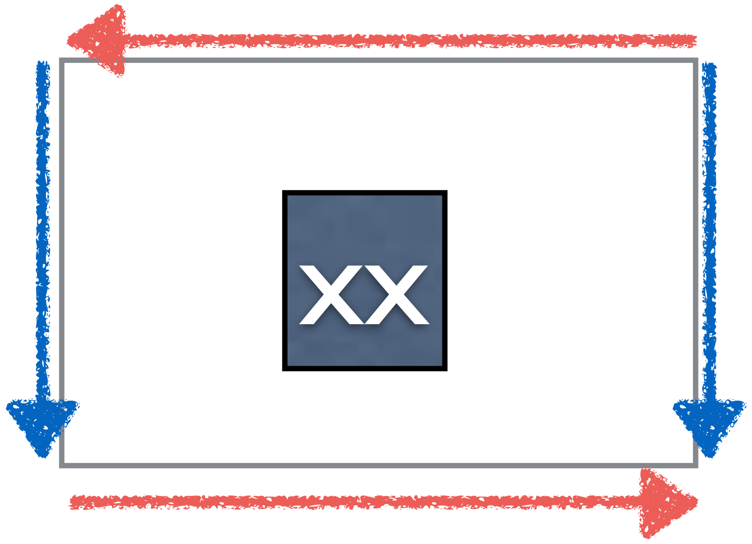


universal cover - H2 tiled by (8,8)!

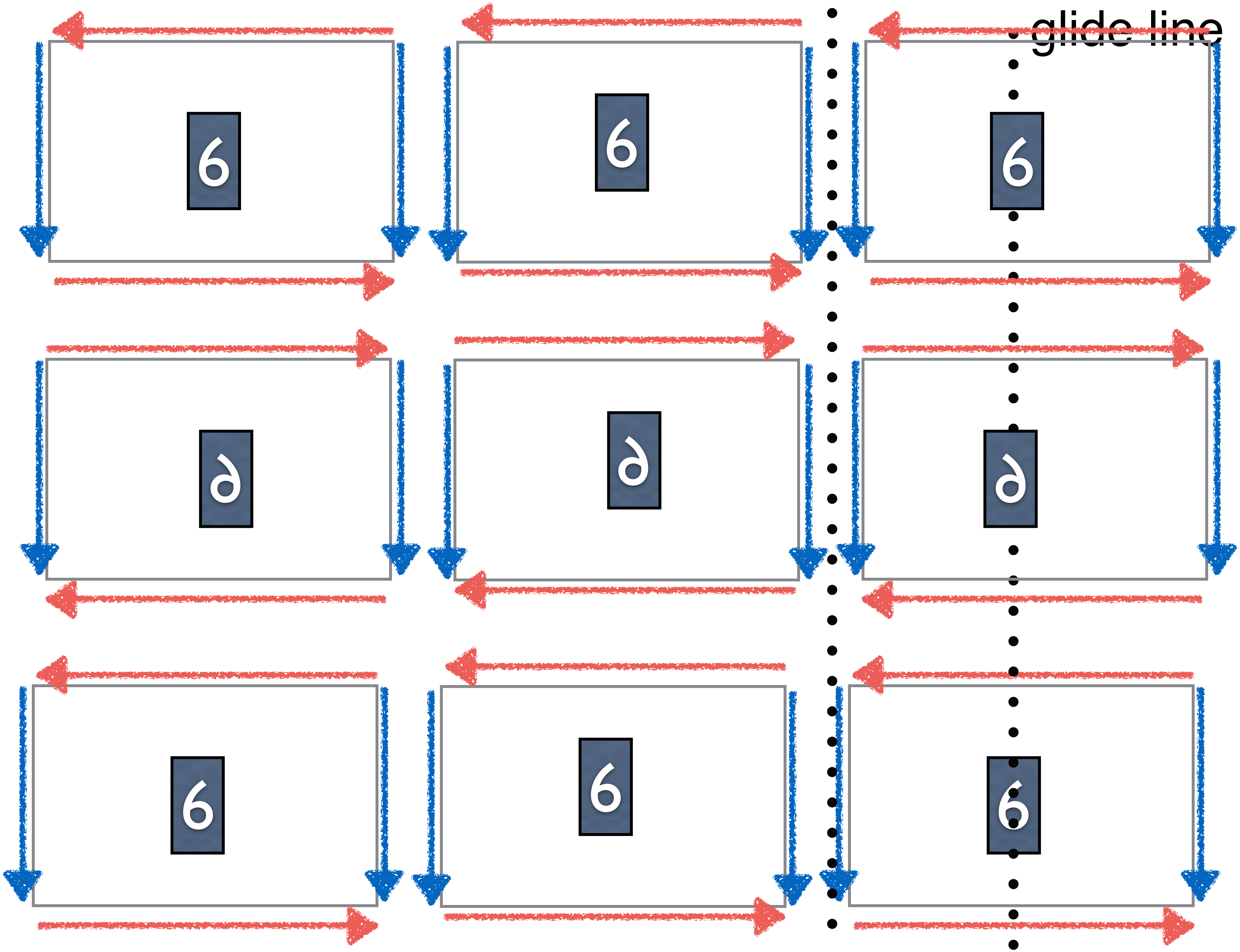


torus

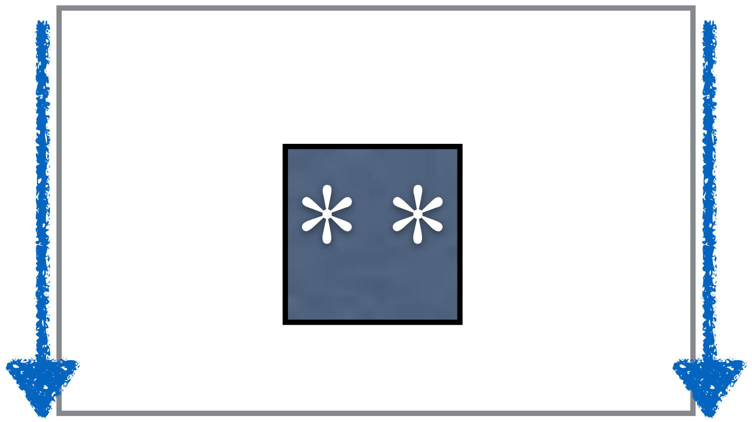
O = p1 wallpaper!



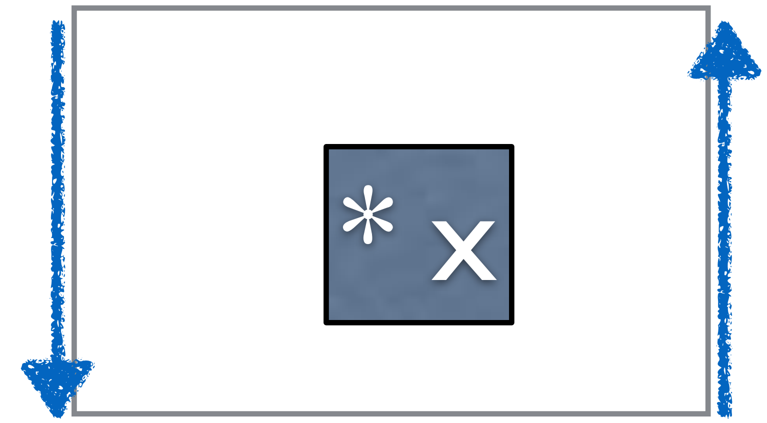
Klein bottle



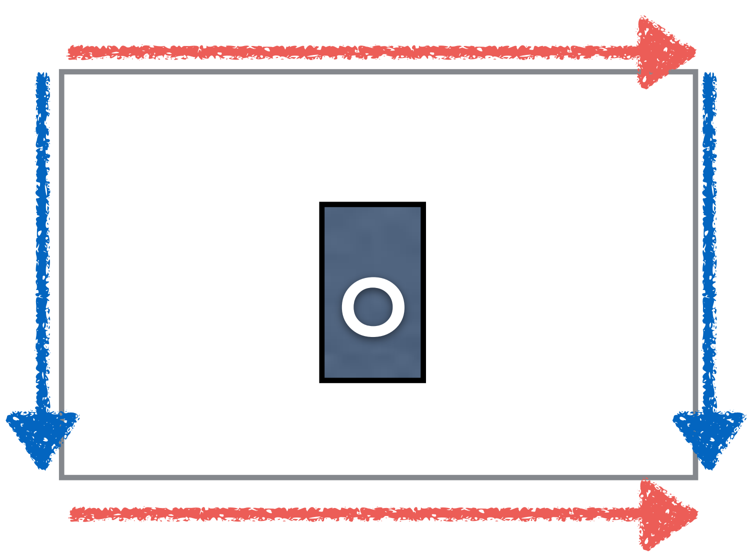
Klein bottle xx = pg wallpaper



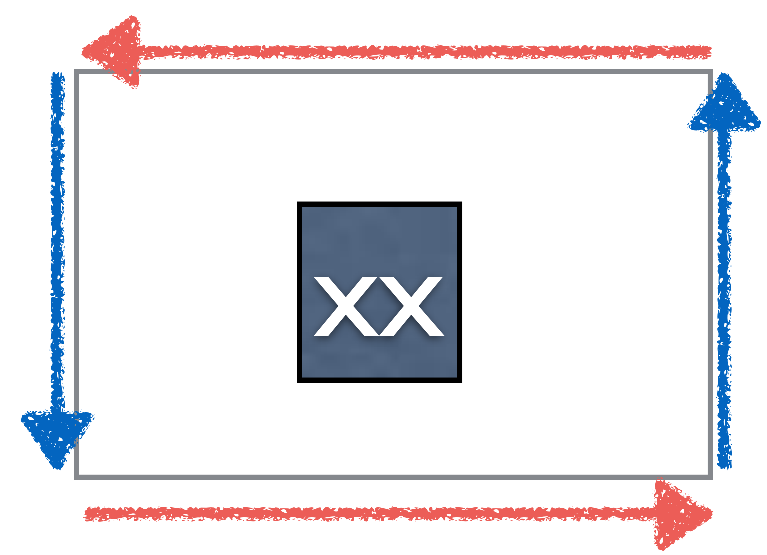
cylinder



Moebius strip

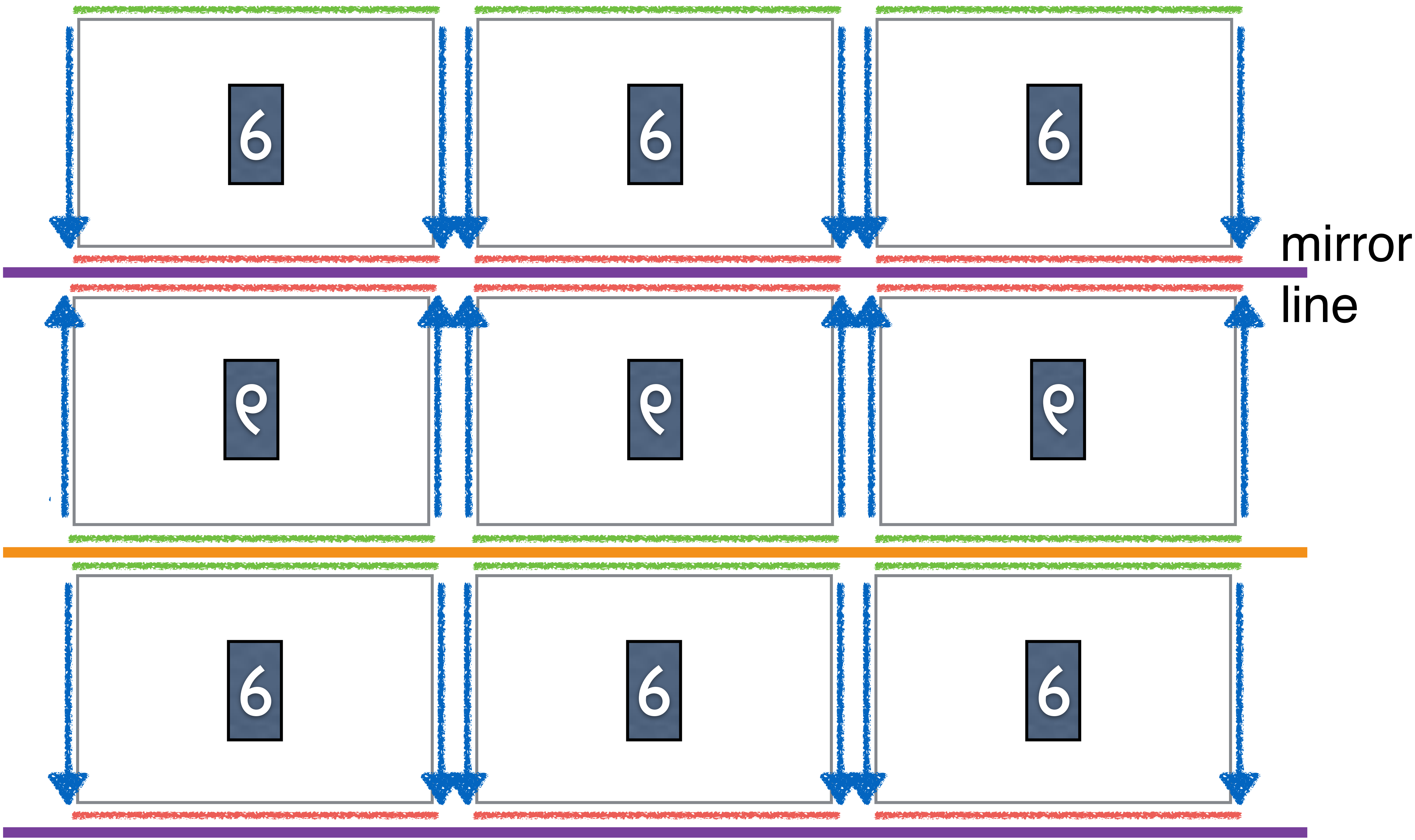


torus



Klein bottle

cylinder



** = pm wallpaper!

Conway symbols describe “orbifolds”

Orbifolds describe symmetric patterns