New Zealand INSTITUTE for Advanced Study

CTCP Centre for Theoretical Chemistry and Physics

Lecture 3: Quantum solitons and beyond

The Dodd-Walls Centre for Photonics and Quantum Technology

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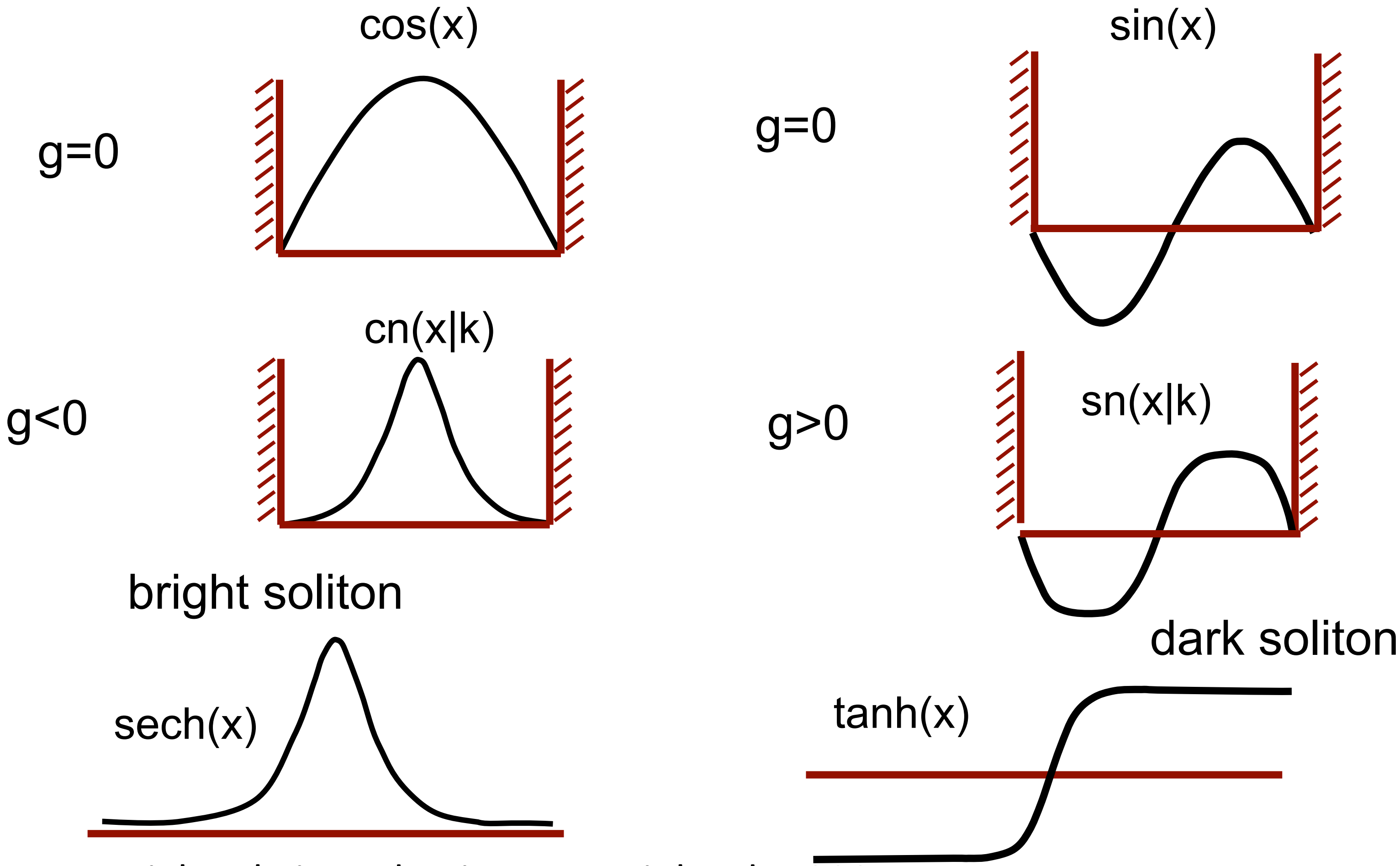
Solitons: Lecture 3

- Plan for today:
- solitary waves in elongated, 3 D geometry
  - Recap of last lecture
  - Chladni solitons
  - Solitonic vortex in image vortex model
- Physics of 1D Bose gas
  - Quantum effects in bright solitons
  - Lieb Liniger model
  - Type II excitations and *quantum* dark solitons

Solitons as stationary solutions of the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}u(x,t) = [-\frac{1}{2}\frac{\partial^2}{\partial x^2} + g|u|^2]u(x,t)$$

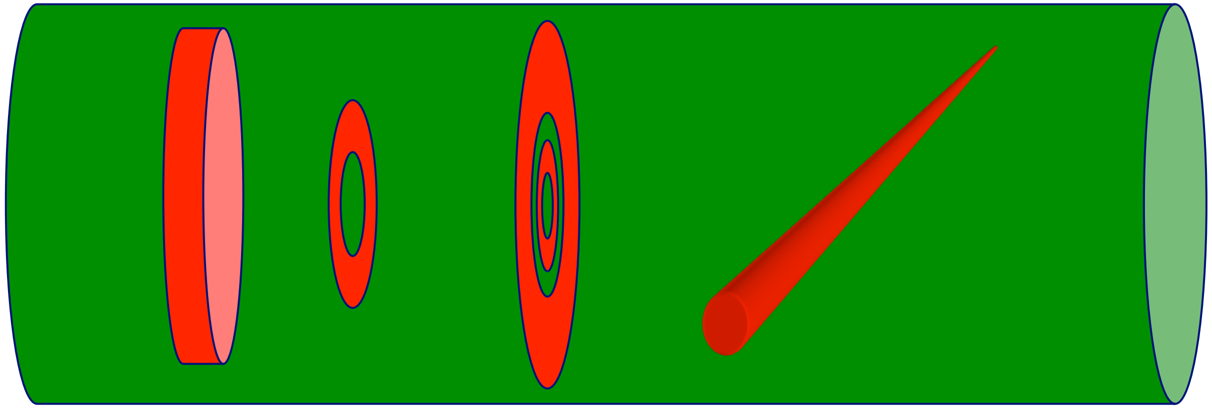


For a tutorial-style introduction see Reinhardt 1988

Solitary waves in 3D waveguides

axially symmetric

not axially symmetric



planar soliton

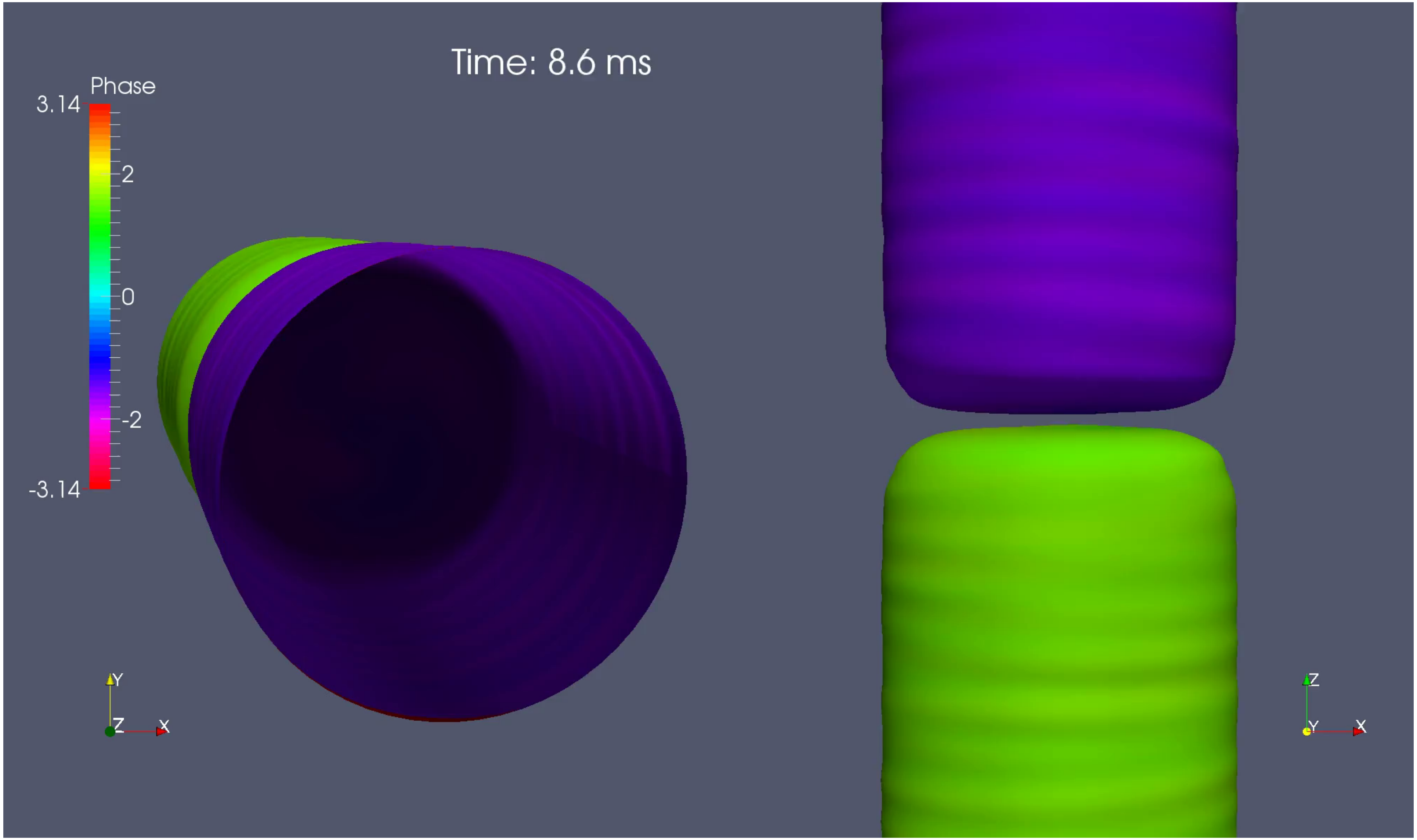
solitonic vortex

vortex ring

double ring

more ...

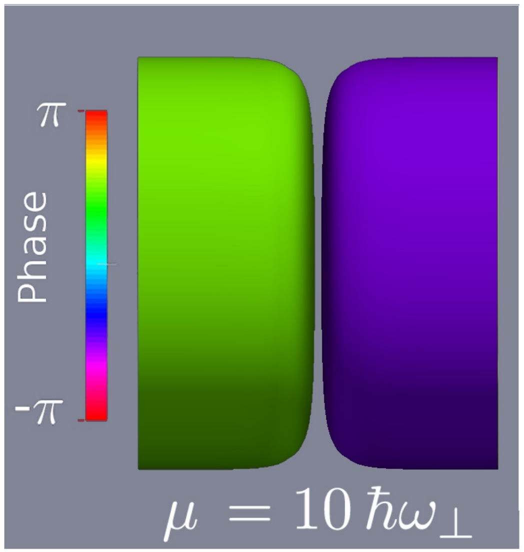
Decay of planar dark soliton



Snaking instability of dark soliton in cylindrical trap?

Hydrodynamic picture of the snaking instability: Dark soliton is a membrane that “vibrates” under the influence of *surface tension* (and negative mass density).

Kamchatnov, Pitaevskii PRL (2008)



Thus, we should expect the vibration modes of a circular membrane …

Entdeckungen

über die

Theorie des Klanges

"Discoveries about the Theory of Chimes"

von

Ernst Florens Friedrich Chladni,

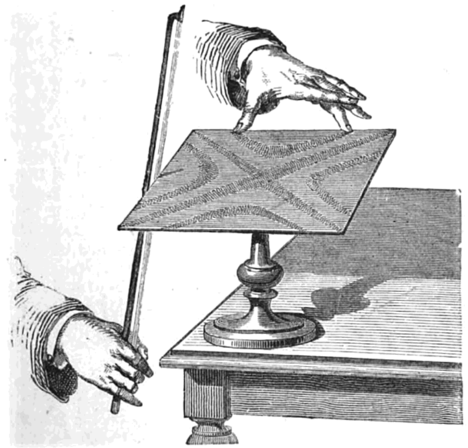
der Philosophie und Rechte Doctor zu Wittenberg.

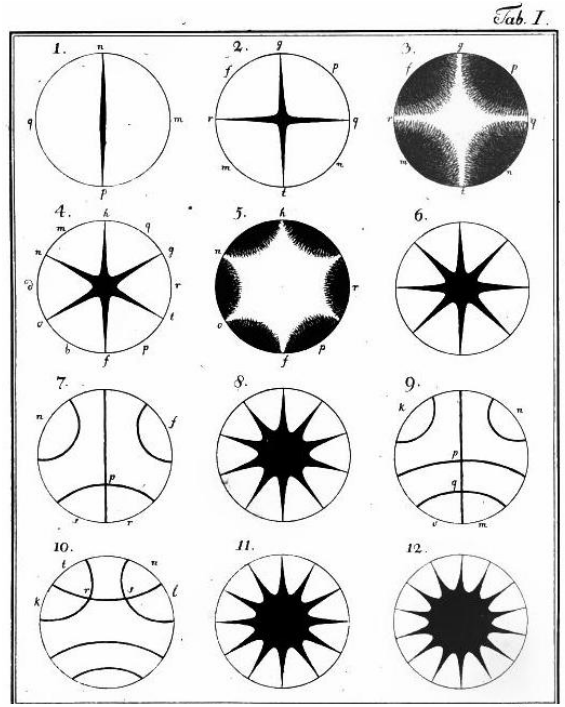
Mit eilf Kupfertafeln.

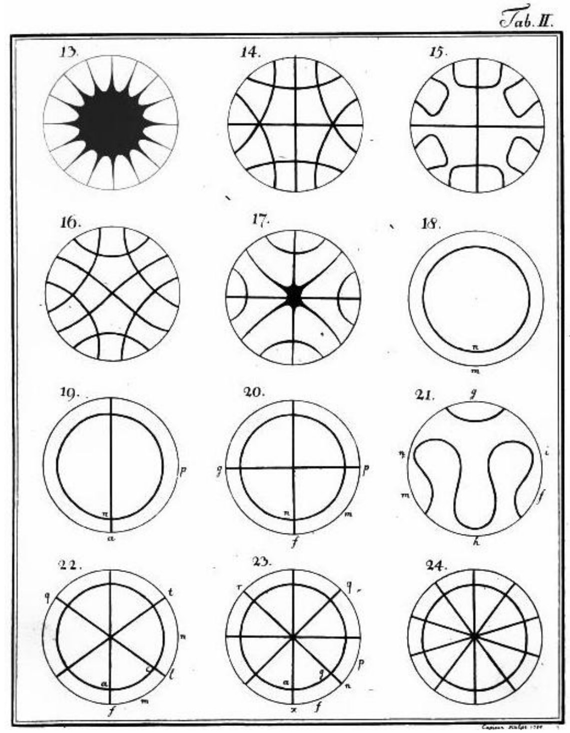
Leipzig,

bey Weidmanns Erben und Reich.

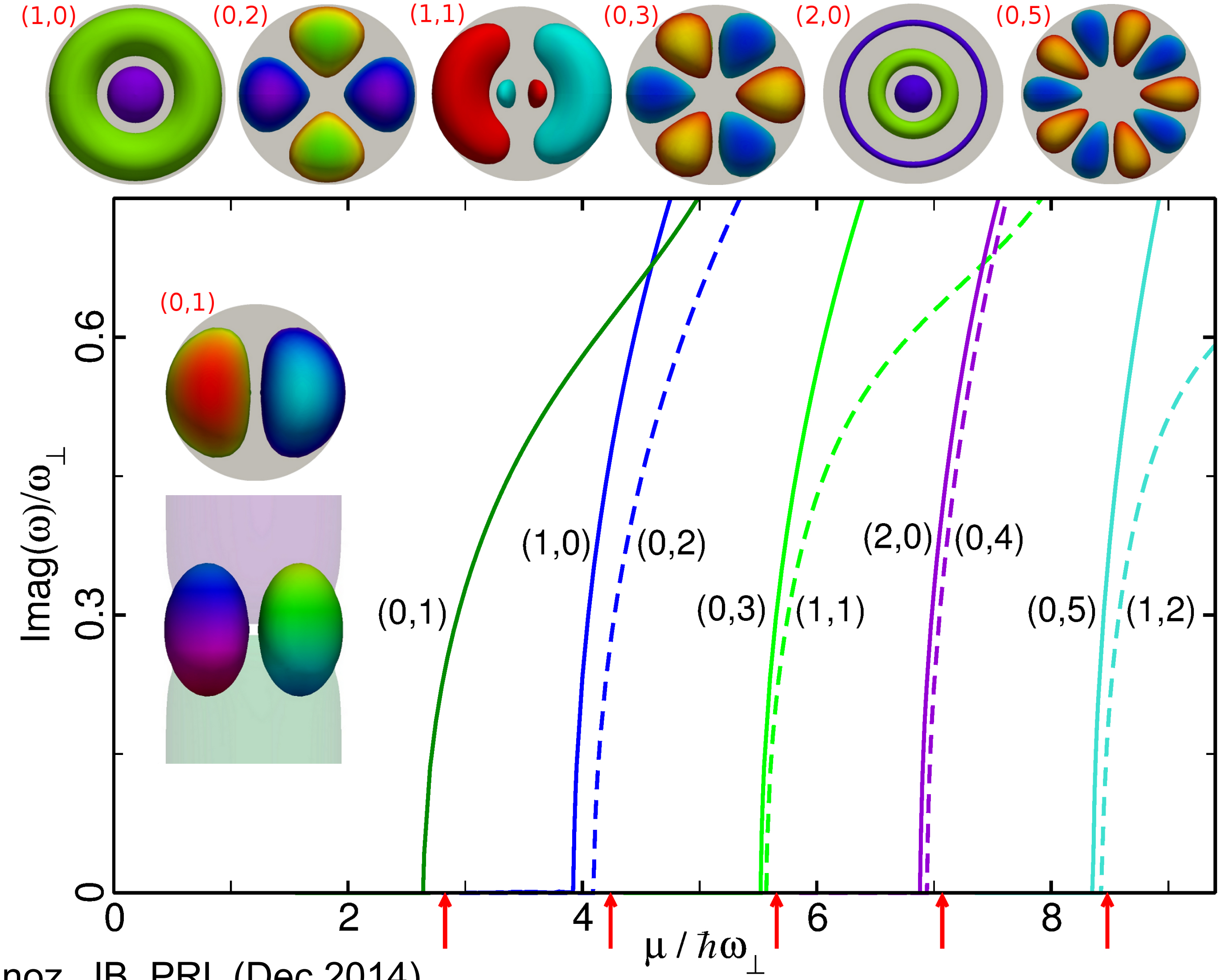
1787.





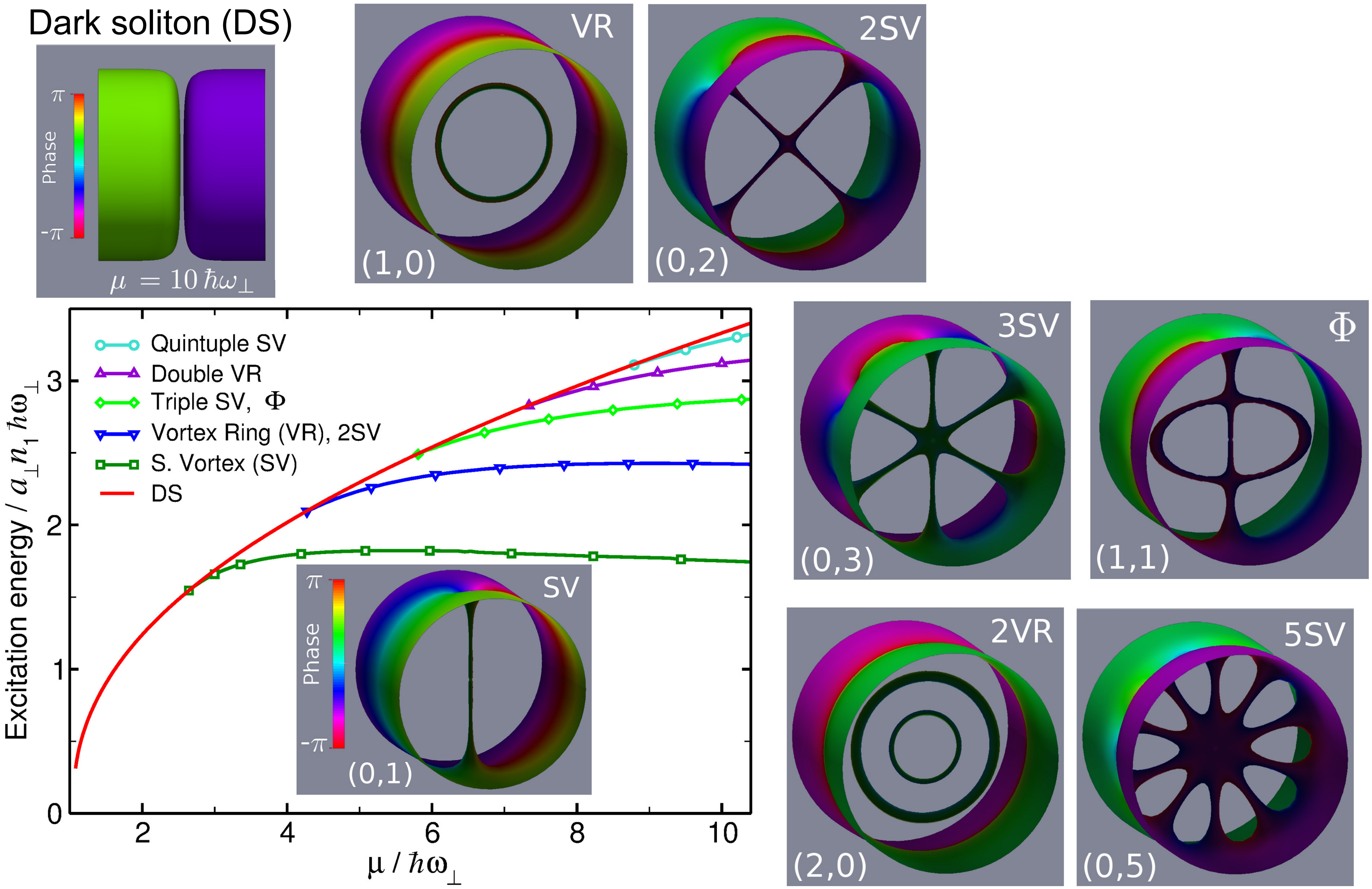


Unstable modes of the dark soliton (numerics)



A. Mateo Munoz, JB, PRL (Dec 2014)

Chladni Solitons: Numerics (GPE)



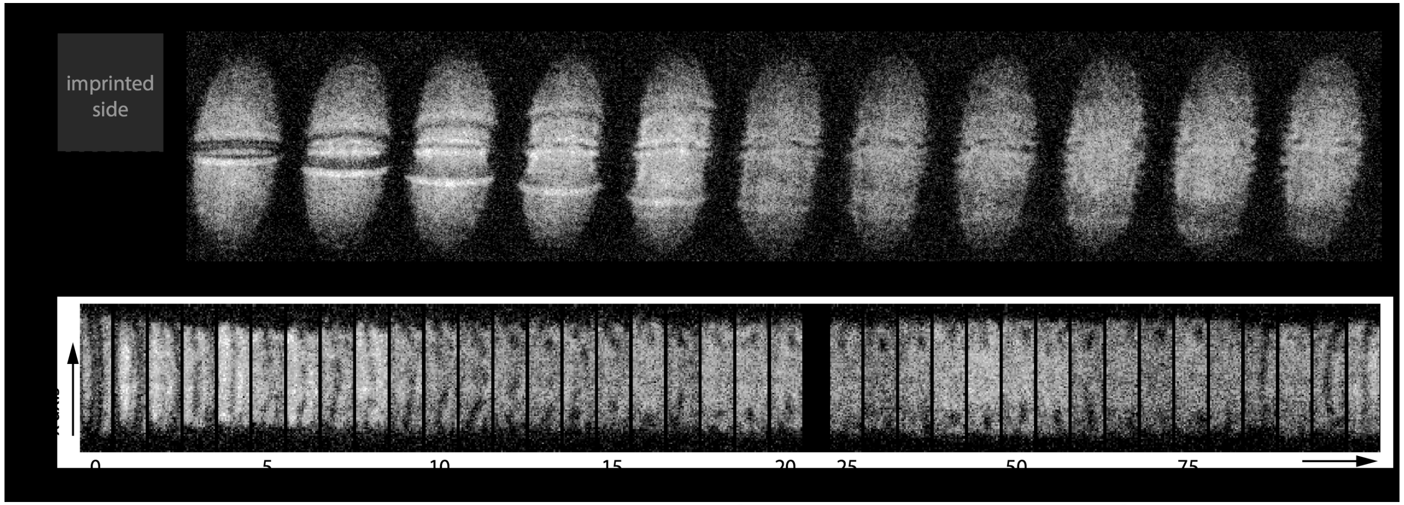
A. Mateo Munoz, JB, PRL (Dec 2014)

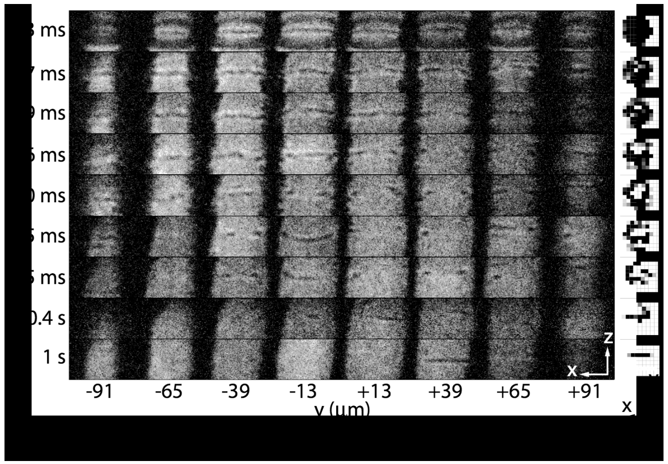
Decay of planar dark solitons observed in the unitary Fermi gas

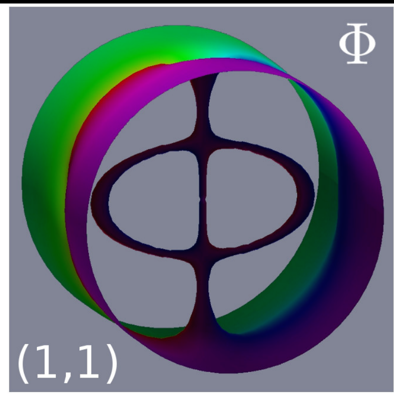
PRL **116,** 045304 (2016) PHYSICAL REVIEW LETTERS week ending 29 JANUARY 2016

Cascade of Solitonic Excitations in a Superfluid Fermi gas: From Planar Solitons to Vortex Rings and Lines

Mark J. H. Ku, Biswaroop Mukherjee, Tarik Yefsah, and Martin W. Zwierlein

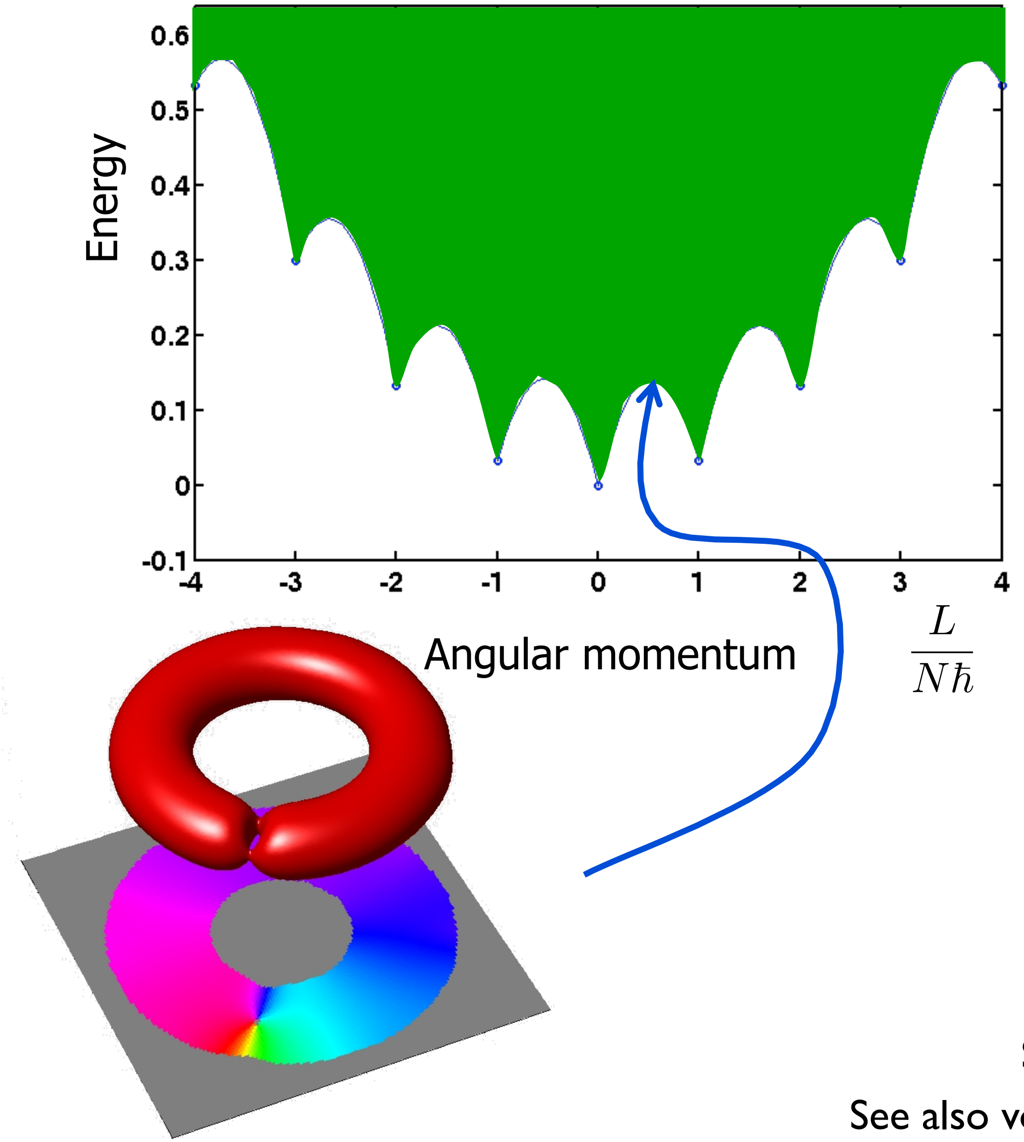






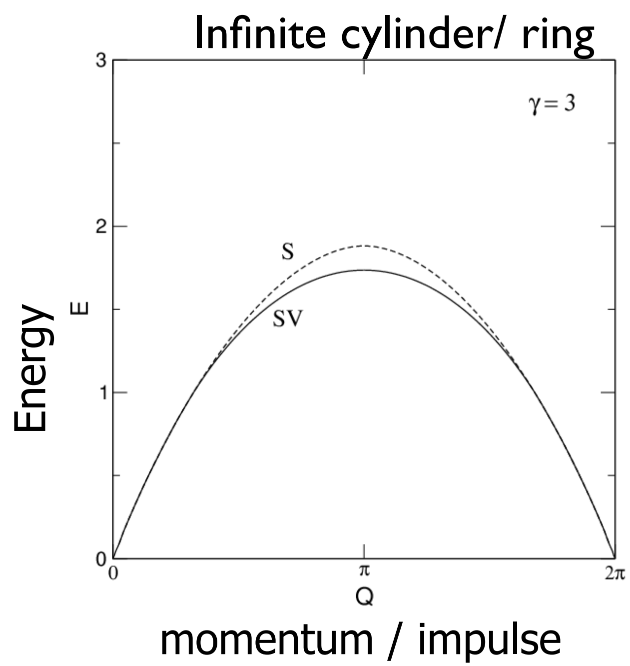
Phi soliton observed

Why is the solitonic vortex stable?
Think ring BEC!



JB, WP Reinhardt, J. Phys. B **37**, L113 (2001)

Solitonic vortices are ***yrast states***, the lowest energy state for given angular momentum.



S. Komineas, N. Papanicolao PRA 68, 043617 (2003)

See also vortex in small anulus:
P. Mason and N. Berloff, PRA 79, 043620 (2009)

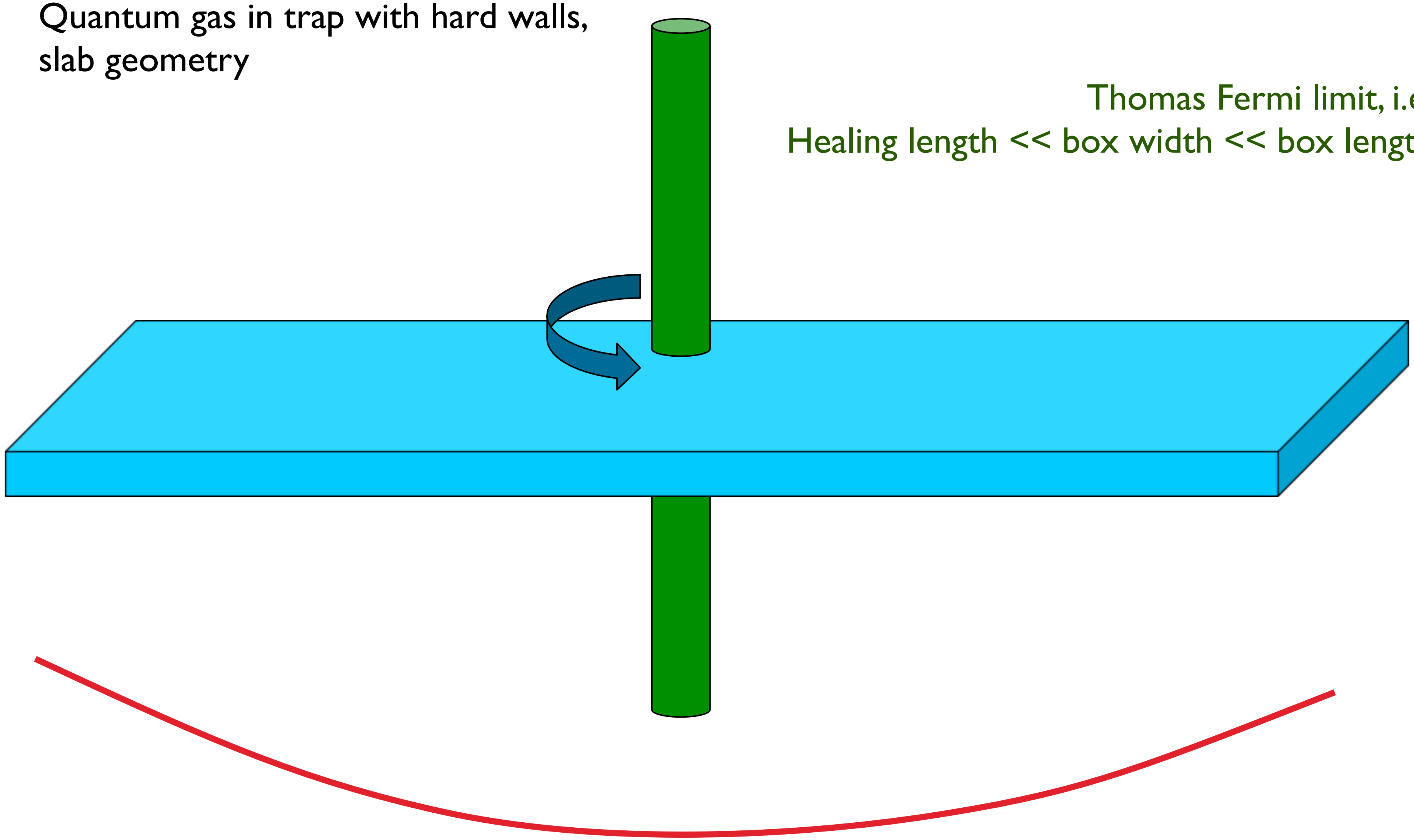
Why would a solitonic vortex oscillate more slowly?

It has a large ratio of effective to physical mass.

Solitonic vortex in a slab geometry

Quantum gas in trap with hard walls,
slab geometry

Thomas Fermi limit, i.e.
Healing length << box width << box length



Weak harmonic potential in long direction

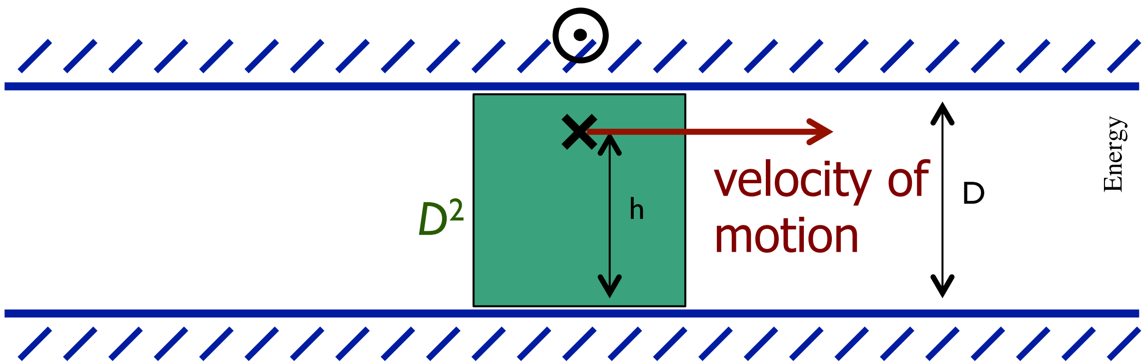
Method of images

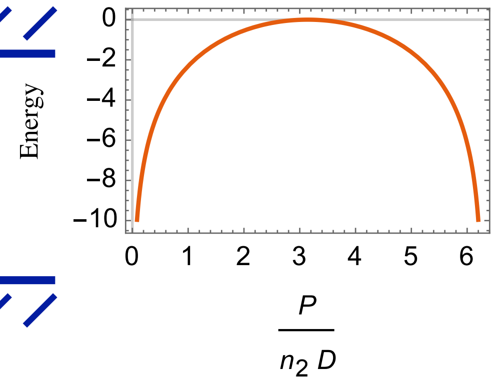
Velocity potential

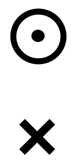
$$w = i \ln \frac{\sinh \frac{\pi}{2D}(z - ih)}{\sinh \frac{\pi}{2D}[z - i(2D - h)]}$$

Energy-momentum dispersion relation

$$E_s = \pi n_2 \ln \sin\left(\frac{P}{2n_2 D}\right)$$







Effective mass (v=0)

$$m^* = -m\frac{4}{\pi}D^2 n_2$$

All particles in volume D^2 contribute to the effective/inertial mass

Solitonic vortex in a slab geometry

Effective mass (v=0)

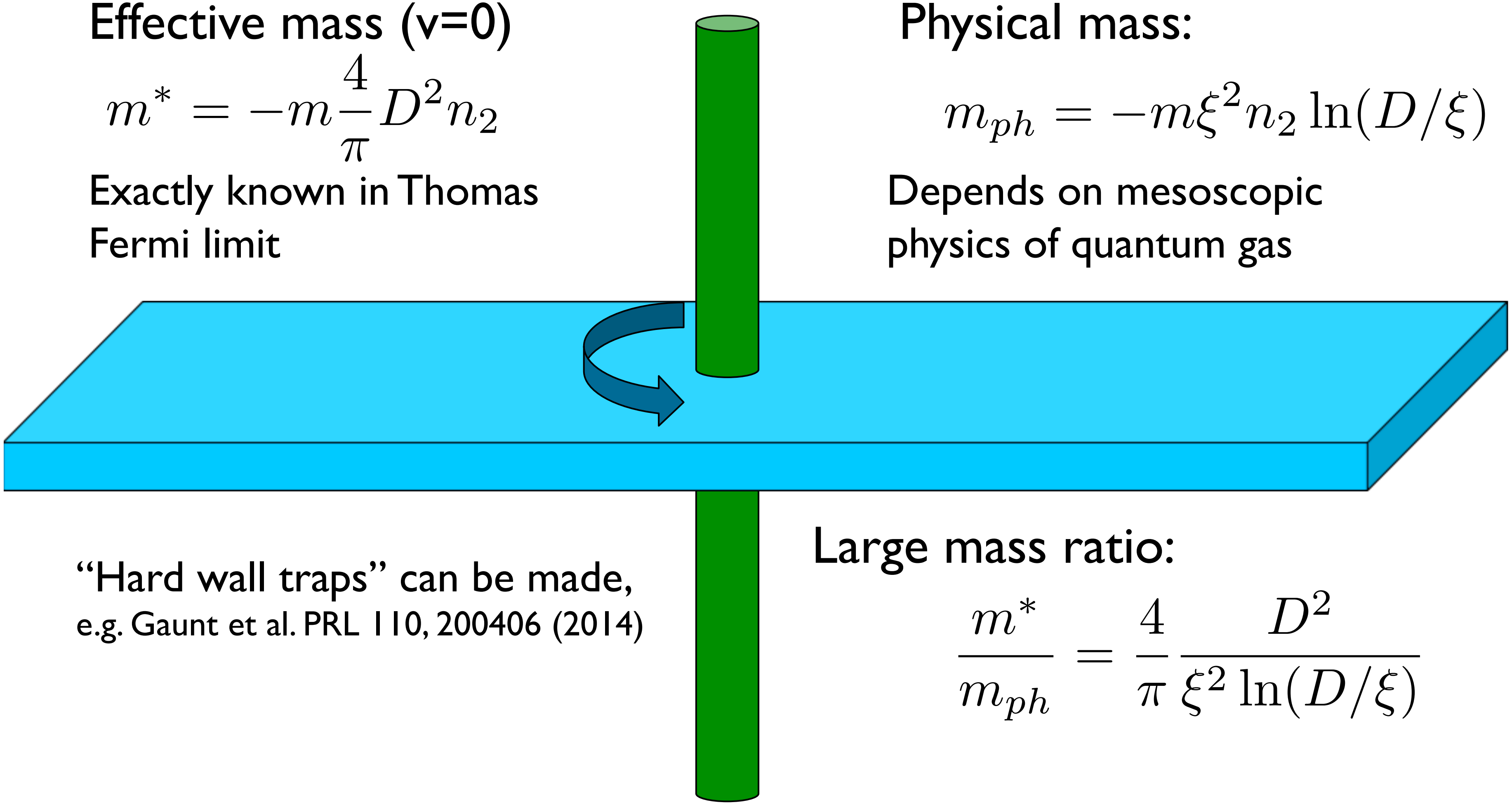
$$m^* = -m\frac{4}{\pi}D^2 n_2$$

Exactly known in Thomas Fermi limit

Physical mass:

$$m_{ph} = -m\xi^2 n_2 \ln(D/\xi)$$

Depends on mesoscopic physics of quantum gas



“Hard wall traps” can be made, e.g. Gaunt et al. PRL 110, 200406 (2014)

Large mass ratio:

$$\frac{m^*}{m_{ph}} = \frac{4}{\pi}\frac{D^2}{\xi^2 \ln(D/\xi)}$$

An experiment measuring the oscillation frequency of a solitonic vortex could measure precisely the filling factor of the vortex core.

Lauri Toikka, JB, *NJP* (2017)

Quantum effects in solitons?

Let’s go to one dimension.

Ground state for *N* attractive bosons in 1D box (Lai, Haus 1989)

Quantum mechanics (Mc Guire 1964)

- Bound state (cluster) of *N* particles
- Non-degenerate
- CoM delocalised quantum particle

GP mean field theory

- $\phi(x) = \mathrm{sech}(x)$
  or $\mathrm{cn}(x|m)$
- Highly degenerate (position of soliton)
- CoM localised classical particle

$$g^2(x-x') = \langle \psi^\dagger(x)\psi^\dagger(x')\psi(x')\psi(x)\rangle \approx \mathrm{sech}^4(x-x')$$

Reality is actually a bit more complicated but in essence the g2 function is bell-shaped in both theories. For a detailed comparison see Kärtner and Haus PRA 48, 2361 (1993).

Quantum description of attractive bosons in 1D

- Exact solutions by J. B. McGuire (1964) for 1D bosons with attractive delta interaction
  - There is exactly one bound state for N particles. This is the ground state
  - All other solutions of N particles are scattering states. The scattering phase shifts can be determined.
- Quantum solitons as superpositions of McGuire bound states (Lai, Haus 1989)
  - Density profile and energies of GPE solitons compares very well with exact solutions
  - Centre of mass motion is that of a free quantum particle with mass *Nm*. Centre of mass should spread ballistically.
- Phase space/field theory treatment of quantum solitons by Drummond/Carter (1987)
  - Predicts squeezing in the number/phase uncertainty
  - Observed in 1991 (Rosenbluh, Shelby), also Leuchs (2001)

Predicted quantum effects

- Cat states
  - Scattering on a sharp barrier at very low kinetic energy should create superposition of soliton going left and soliton going right (Schrödinger cat state).
    Weiss and Castin (2009)
  - Quantum motion of the centre of mass (hard)
- Dissociation
  - Upon a sudden increase of interaction strength a splitting-up of the soliton into multiple fragments could be observed.
    Yurovsky, Malomed, Hulet, Olshanii (2017)

Quantum effects in dark solitons?

1D physics is described by the Lieb Liniger model.

1D Bose Gas – Lieb-Liniger model

$$H = \sum_i \frac{p_i^2}{2m} + 2g_{1\mathrm{D}} \sum_{i<j} \delta(x_i - x_j)$$

- 1D Bosons with repulsive δ interactions
- Ground- and excited-state wavefunctions exactly known from Bethe ansatz [Lieb, Liniger (1963)]
- Interaction parameter $\gamma = \frac{m}{\hbar^2} \frac{g_{1\mathrm{D}}}{n}$
- For $\gamma \to \infty$, problem is mapped exactly to free Fermi gas (Tonks-Girardeau gas) [Girardeau (1960)]
- Ring geometry provides periodic boundary conditions

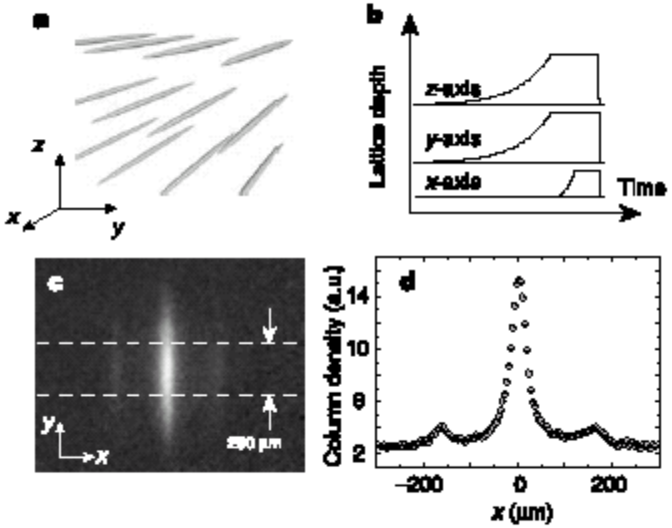
Tonks-gas – Experiments

letters to nature

Tonks–Girardeau gas of ultracold atoms in an optical lattice

Belén Paredes[1], Artur Widera[1,2,3], Valentin Murg[1], Olaf Mandel[1,2,3], Simon Fölling[1,2,3], Ignacio Cirac[1], Gora V. Shlyapnikov[4], Theodor W. Hänsch[1,2] & Immanuel Bloch[1,2,3]

MPQ Garching (2004)



up to $\gamma_{\mathrm{eff}} \sim 200$

other experiments:

T. Esslinger (Zürich)

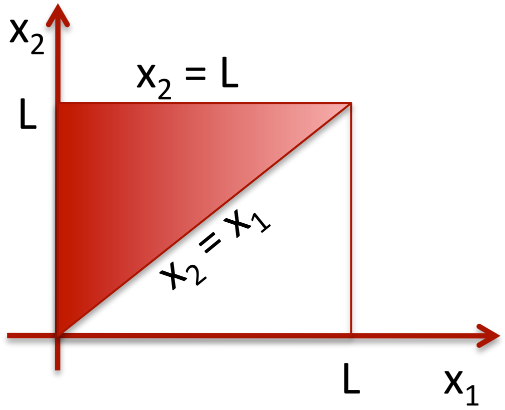
W. Phillips (NIST)

D. Weiss (PSU), $\gamma \sim 5.5$

R. Grimm (Innsbruck): confinement induced resonance!

$$\gamma \approx \frac{\text{interaction energy}}{\text{kinetic energy}}$$

Lieb-Liniger model: wave function



Consider $0 \leq x_1 \leq x_2 \cdots \leq x_N \leq L$

- Inside: $$-\frac{\hbar^2}{2m}\sum_i \frac{\partial^2}{\partial x_i^2}\psi = E\psi$$
- Boundary conditions are provided by
  - Interactions
  - Periodicity in the box
- Bethe ansatz:

$$\psi(x_1, \ldots, x_N) = \sum_P a(P) P \exp(i \sum_{j=1}^{N} k_j x_j)$$

P is a permutation of the set $\{k\} = k_1, k_2, \ldots, k_N$

- Just one quasimomentum per particle (!)
- Model is integrable, check Yang-Baxter equation

Bethe ansatz equations

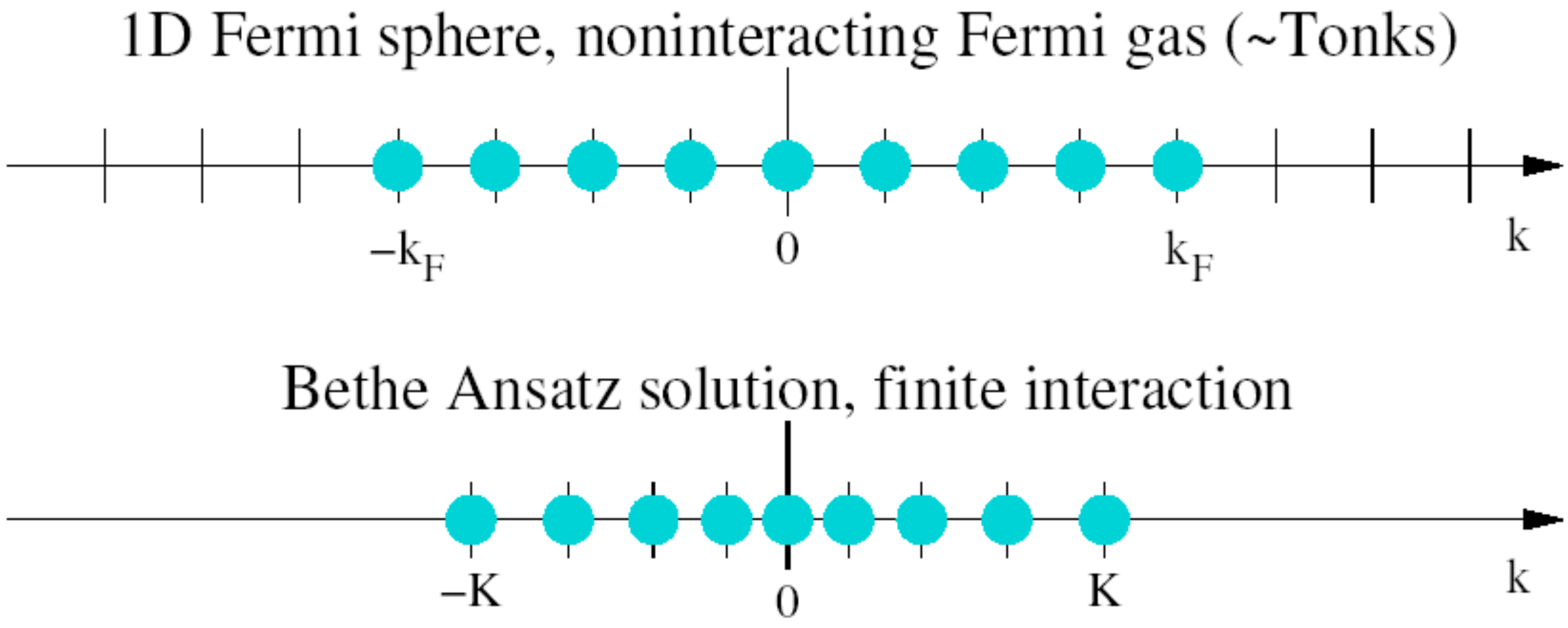
$$k_j + \frac{1}{L}\sum_{l=1}^{N} 2\arctan\frac{k_j - k_l}{c} = \frac{2\pi}{L} I_j$$

k_j - charge rapidities
I_j - integer (half-integer) valued quantum numbers
N - number of bosons
L - length of periodic box

$$P_{\text{tot}} = \hbar \sum_{j=1}^{N} k_j,$$

$$E_{\text{tot}} = \frac{\hbar^2}{2m}\sum_{j=1}^{N} k_j^2.$$

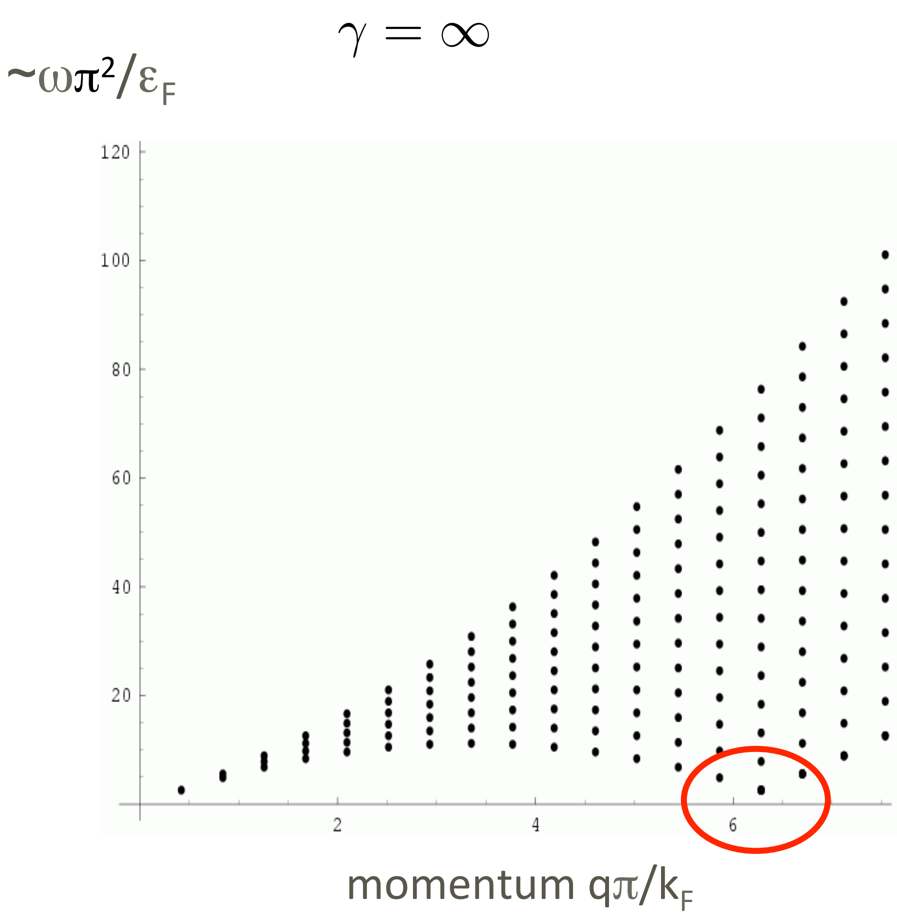
The nature of Bethe-ansatz solutions: Quasi-momenta and Fermi sphere

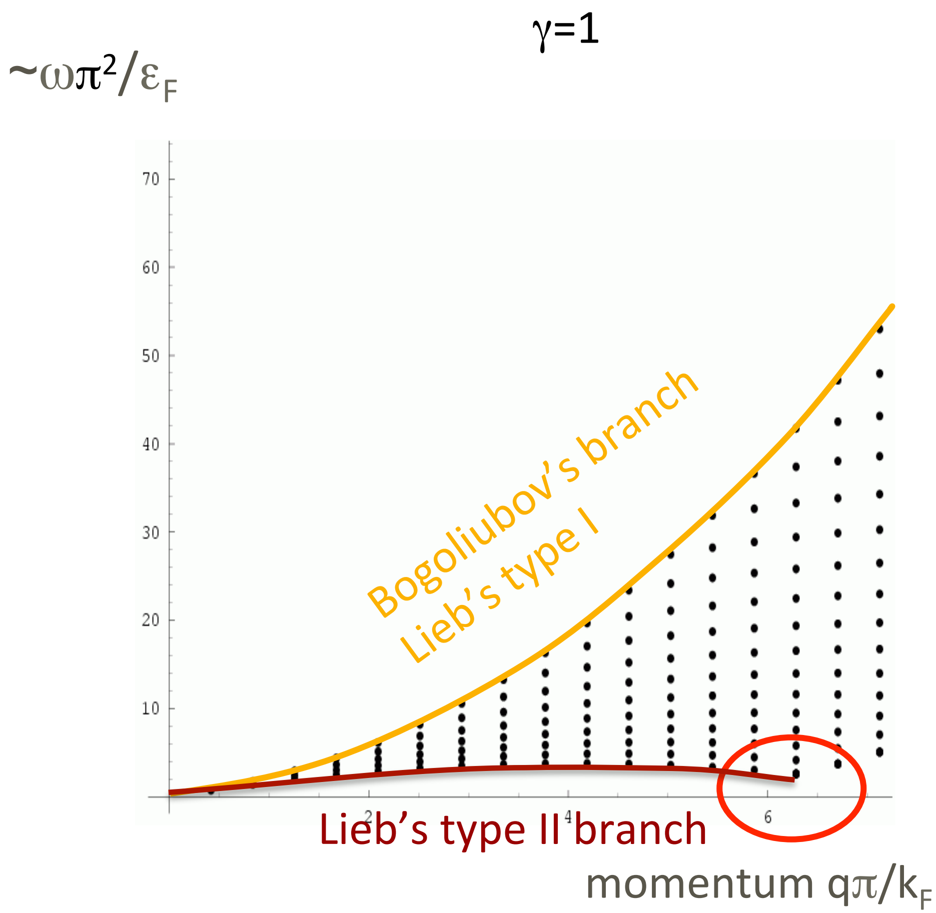


Total energy: $E = \frac{\hbar^2}{2m} \sum_k k^2$

Total momentum: $P = \hbar \sum_k k$

Excitation spectrum for the Lieb-Liniger model



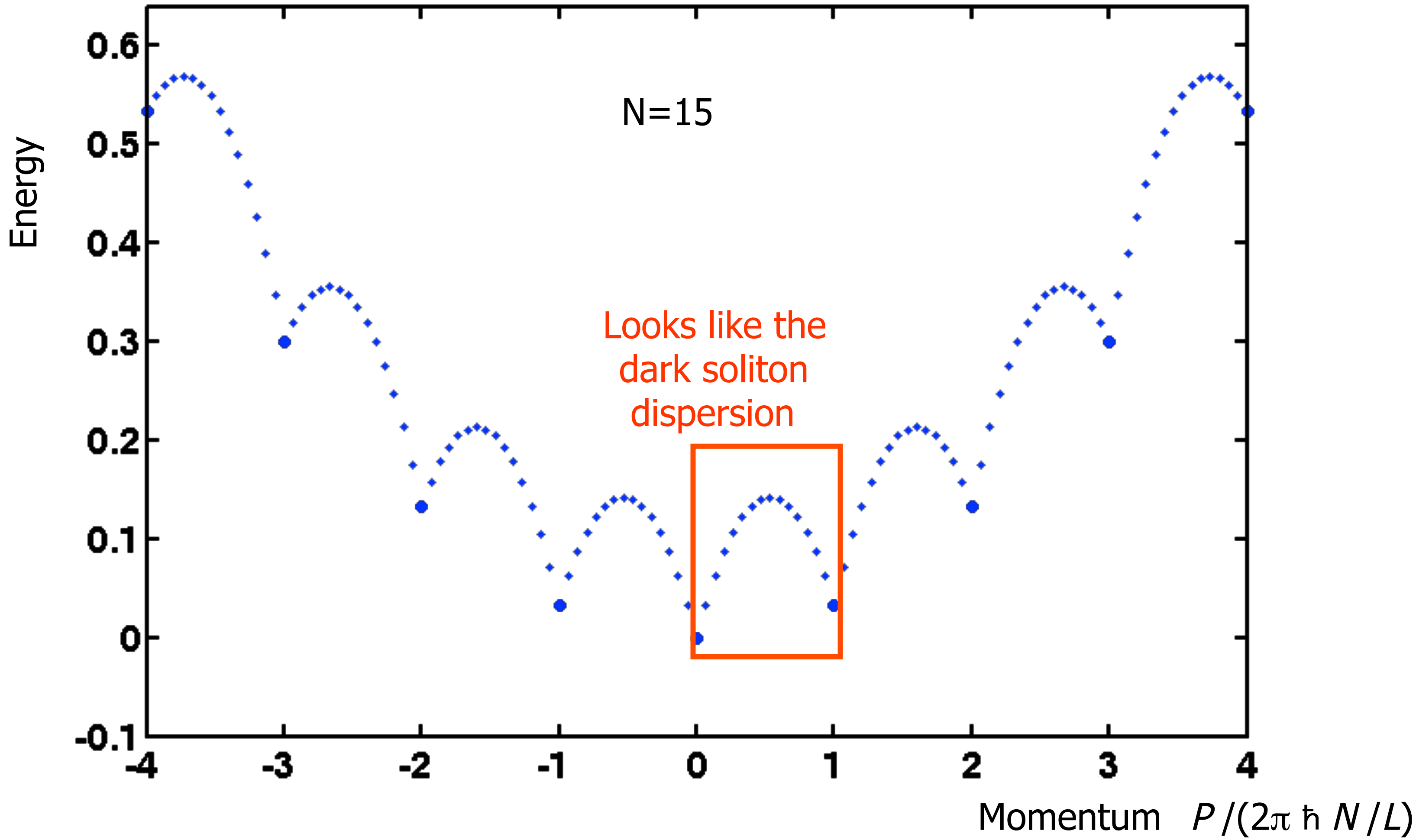


umklapp excitation

$k_F = \pi n_{1D}$; $\varepsilon_F = \hbar^2 k_F^2/(2m)$

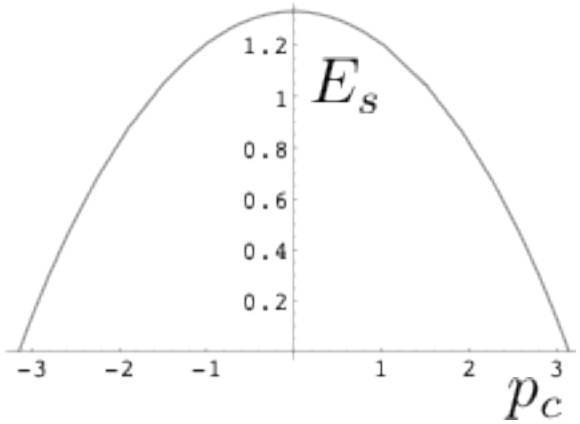
Type II excitations can be identified with dark solitons!

Low-lying excitation spectrum (yrast states)

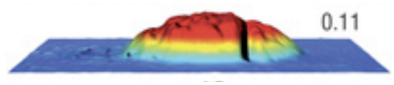


Soliton dispersion

- Soliton energy: $E_s(\mu, v_s, g) = \langle \hat{H} - \mu \hat{N} \rangle - E_h$
- Canonical momentum: $v_s = \dfrac{dE_s}{dp_c}$
- Effective (inertial) mass: $m^* = 2\dfrac{\partial E_s}{\partial (v_s)^2}$
- Physical (heavy) mass: $m_{ph} = mN_s$



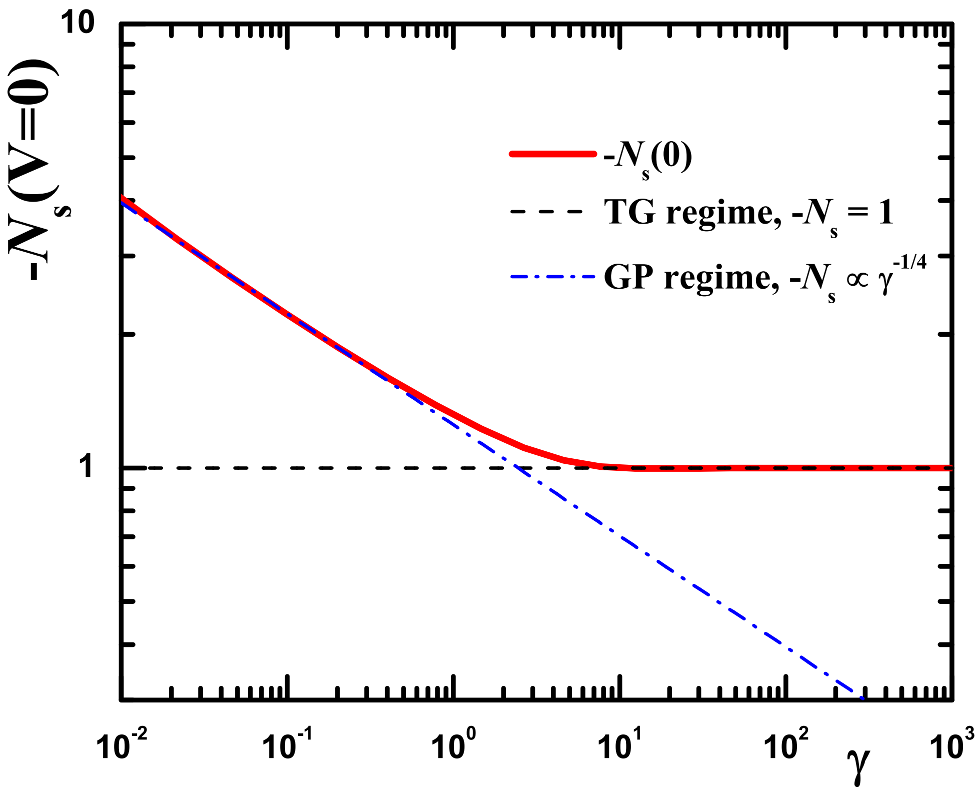
$$N_s = \int (n_s - n_0) d^3 r = -\frac{\partial E_s}{\partial \mu}$$ (for $v = 0$)



The dark soliton dispersion (in the right units) asymptotically matches the Lieb type II dispersion relation for large densities. Ishikawa, Takayama JPSJ (1980)

So: We can use the dispersion relation to calculate properties of the "quantum dark soliton" in the Lieb-Liniger model.

Dark soliton particle number (missing particles) in Lieb-Liniger gas



Weakly interacting 1D Bose gas,
GP description holds

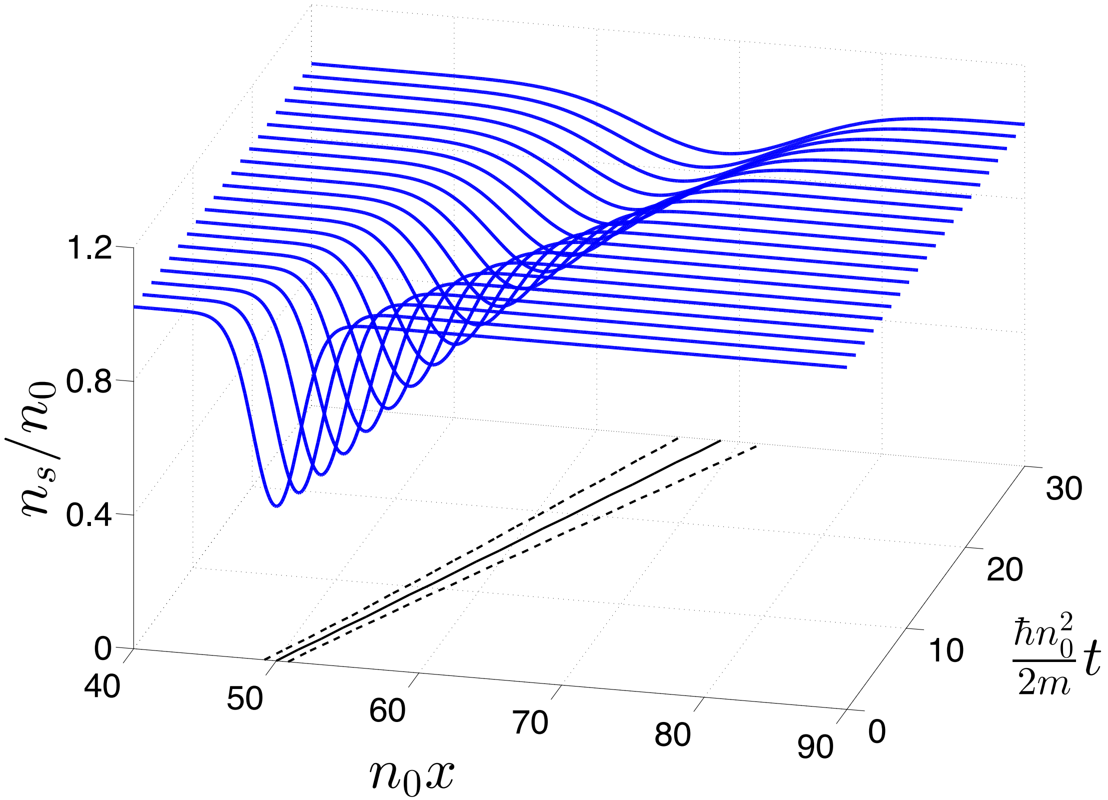
Tonks-Girardeau gas limit,
A quantum dark soliton has
one missing particle

Astrakharchik, Pitaevskii (2012)

But how to obtain a solitary wave?

- Construct a **quantum dark soliton** as a superposition of yrast states
- Density profile can be obtained with formulas from the algebraic Bethe ansatz

- Localised density depression propagates at constant velcocity and spreads over time
- Strongly analogous to quantum bright soliton



Shamailov, Brand, *unpublished*