Introduction to Integrability in AdS/CFT: Lecture 3

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Introduction

Recall:

- 1-loop anomalous dimensions in SU(2) subsector of $\mathcal{N}$=4 SYM are given by Bethe ansatz

key to solution: R-matrix $\Rightarrow$ commuting transfer matrix $\cup$ Hamiltonian

Would like to understand:

- other operators? $\longleftarrow$ other R-matrices!
- higher loops? $\longleftarrow$ R-matrix approach seems not to work !?

Alternative approach: S-matrix & coordinate Bethe ansatz!

(Techniques from previous lecture still useful!)

Plan

- Today: still 1 loop
    - SO(6) scalar sector
    - all (single-trace) operators at 1 loop
    - S-matrix & coordinate Bethe ansatz
    - higher-rank generalization
- Subsequent: higher loops

SO(6) sector

[Minahan & Zarembo 02]

1-loop mixing matrix for $\mathcal{O}[\psi] \equiv \psi_{I_1 \ldots I_L} \operatorname{tr} \phi^{I_1}(x) \ldots \phi^{I_L}(x)$

$$\Gamma = \frac{\lambda}{8\pi^2} H \,, \quad H = \frac{1}{2} \sum_{l=1}^{L} (K_{l,l+1} + 2 - 2\mathcal{P}_{l,l+1}) \qquad L+1 \equiv 1$$

SO(6)-invariant R-matrix:

[Zamolodchikov[2] 79]

$$R(u) = \frac{1}{2} [uK + u(u-2)I - (u-2)\mathcal{P}]$$

Acts on $V^6 \otimes V^6$ (vector reps) Obeys YBE

monodromy matrix: $T_0(u) = R_{0L}(u) \cdots R_{01}(u)$

transfer matrix: $t(u) = \operatorname{tr}_0 T_0(u)$ $\quad [t(u)\,, t(u')] = 0$

$$H \sim \frac{d}{du} \ln t(u) \Big|_{u=0}$$

integrable!

$$t(u)\,|\Lambda\rangle = \Lambda(u)\,|\Lambda\rangle$$

Eigenvalues?

Analytical Bethe ansatz [Reshetikhin 83]

Vacuum: all spins "up"

$$\Lambda_0(u) = \frac{1}{2^L}\Big[(u-2)^L(u-1)^L + (u-1)^L u^L + 4(u-2)^L u^L\Big]$$

Assume general eigenvalue is "dressed" vacuum eigenvalue:

$$\Lambda(u) = \frac{1}{2^L}\Big\{(u-2)^L(u-1)^L\,\frac{Q_1(u+\frac{1}{2})}{Q_1(u-\frac{1}{2})} + (u-1)^L u^L\,\frac{Q_1(u-\frac{5}{2})}{Q_1(u-\frac{3}{2})}$$

$$+(u-2)^L u^L\Big[\frac{Q_1(u-\frac{3}{2})Q_2(u)Q_3(u)}{Q_1(u-\frac{1}{2})Q_2(u-1)Q_3(u-1)} + \frac{Q_1(u-\frac{1}{2})Q_2(u-2)Q_3(u-2)}{Q_1(u-\frac{3}{2})Q_2(u-1)Q_3(u-1)}$$

$$+\frac{Q_2(u)Q_3(u-2)}{Q_2(u-1)Q_3(u-1)} + \frac{Q_2(u-2)Q_3(u)}{Q_2(u-1)Q_3(u-1)}\Big]\Big\}$$

$$Q_a(u) = \prod_{k=1}^{M_a} (u - iu_{a,k}), \quad a = 1,2,3$$

zeros $u_{a,k}$ still to be determined

$\Lambda(u)$ must not have poles $\Rightarrow$ Bethe equations

$$e_1(u_{1,k})^L = \prod_{\substack{j=1 \\ j\neq k}}^{M_1} e_2(u_{1,k} - u_{1,j}) \prod_{j=1}^{M_2} e_{-1}(u_{1,k} - u_{2,j}) \prod_{j=1}^{M_3} e_{-1}(u_{1,k} - u_{3,j})$$

$$1 = \prod_{\substack{j=1 \\ j\neq k}}^{M_2} e_2(u_{2,k} - u_{2,j}) \prod_{j=1}^{M_1} e_{-1}(u_{2,k} - u_{1,j})$$

$$1 = \prod_{\substack{j=1 \\ j\neq k}}^{M_3} e_2(u_{3,k} - u_{3,j}) \prod_{j=1}^{M_1} e_{-1}(u_{3,k} - u_{1,j})$$

$$e_n(u) = \frac{u + in/2}{u - in/2} \qquad \gamma \propto \frac{d}{du} \ln \Lambda(u)\Big|_{u=0} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M_1} \frac{1}{u_{1,k}^2 + \frac{1}{4}}$$

$$Q_a(u) = \prod_{k=1}^{M_a} (u - i u_{a,k})\,, \qquad a = 1,2,3$$

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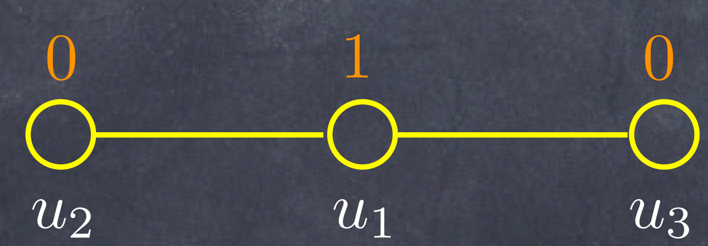
Cartan matrix for so(6) = su(4):

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

$$e_n(u) = \frac{u + in/2}{u - in/2}$$



[Ogievetsky & Wiegmann 86, ...]

Lie (super)algebra symmetry & representation content $\Rightarrow$ Bethe equations!

$$e_{V_l}(u_{l,j})^L = \prod_{\substack{k=1\\k\neq j}}^{M_l} e_{A_{l,l}}(u_{l,j}-u_{l,k}) \prod_{l'\neq l} \prod_{k=1}^{M_{l'}} e_{A_{l,l'}}(u_{l,j}-u_{l',k})$$

$A_{l,l'}$ Cartan matrix of Lie (super)algebra

V_l Dynkin labels ("coefficients") of representation

[review: Cahn 84]

cyclicity $\Rightarrow$

$$\gamma = \frac{\lambda}{8\pi^2} \sum_l \sum_{k=1}^{M_l} \frac{V_l}{u_{l,k}^2 + \frac{1}{4}{V_l}^2}$$

$$P = \frac{1}{i} \sum_l \sum_{k=1}^{M_l} \ln e_{V_l}(u_{l,k}) = 0$$

All operators at 1 loop

$$\mathrm{tr}\, O_1(x)O_2(x)\cdots O_L(x)\,, \quad O_i \in \{D^n\phi\,, D^n\psi\,, D^n F\}$$

fundamental rep of PSU(2,2|4)

infinite dimensional!

- 1-loop dilatation operator determined [Beisert 03]
- Can be derived from R-matrix $\therefore$ integrable!

PSU(2,2|4) symmetry $\Rightarrow$ Bethe equations

[Beisert & Staudacher 03]

rank 7 $\Rightarrow$ 7 BEs

superalgebra $\Rightarrow$ Cartan matrix not unique $\Rightarrow$ BEs not unique

physically: different choices of vacuum

$$1=\prod_{k}^{M_2} e_1(u_{1,j}-u_{2,k})$$

$$1=\prod_{k}^{M_1} e_1(u_{2,j}-u_{1,k})\prod_{j\neq k}^{M_2} e_{-2}(u_{2,j}-u_{2,k})\prod_{k}^{M_3} e_1(u_{2,j}-u_{3,k})$$

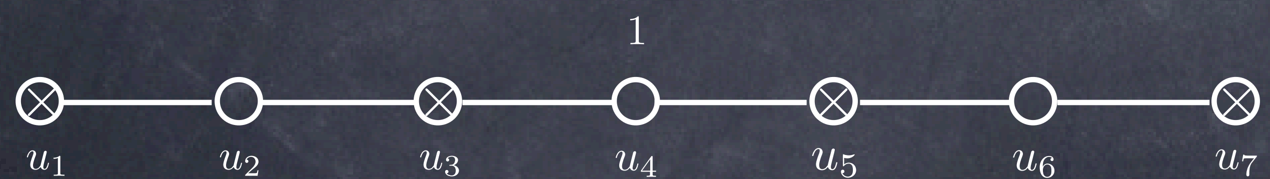
$$1=\prod_{k}^{M_2} e_1(u_{3,j}-u_{2,k})\prod_{k}^{M_4} e_{-1}(u_{3,j}-u_{4,k})$$

$$e_1(u_{4,j})^L=\prod_{k}^{M_3} e_{-1}(u_{4,j}-u_{3,k})\prod_{k\neq j}^{M_4} e_2(u_{4,j}-u_{4,k})\prod_{k}^{M_5} e_{-1}(u_{4,j}-u_{5,k})$$

$$1=\prod_{k}^{M_4} e_{-1}(u_{5,j}-u_{4,k})\prod_{k}^{M_6} e_1(u_{5,j}-u_{6,k})$$

$$1=\prod_{k}^{M_5} e_1(u_{6,j}-u_{5,k})\prod_{j\neq k}^{M_6} e_{-2}(u_{6,j}-u_{6,k})\prod_{k}^{M_7} e_1(u_{6,j}-u_{7,k})$$

$$1=\prod_{k}^{M_6} e_1(u_{7,j}-u_{6,k})$$



S-matrix
&
coordinate Bethe ansatz

As we shall see, at higher loops,
this R-matrix approach does not seem to work.

Another approach,
that does not assume knowledge of R-matrix,
is needed.

Fortunately, such an approach was already known:
the one originally discovered by Bethe!

So, we now return to SU(2)-invariant chain,
and solve it in a different way
"coordinate Bethe ansatz"

key new ingredient: S-matrix

$$H = \sum_{n=1}^{L} (I - \mathcal{P}_{n,n+1})$$ PBCs $L + 1 \equiv 1$

$\{H \,, P \,, S^z\}$ pairwise commute $\therefore$ can diagonalize simultaneously

$$H|\psi\rangle = E|\psi\rangle \,, \quad P|\psi\rangle = P|\psi\rangle \,, \quad S^z|\psi\rangle = m|\psi\rangle$$

vacuum state (all L spins up): $|0\rangle = |Z \cdots Z\rangle$

$\mathcal{P}_{n,n+1}$ interchanges spins at sites n $\leftrightarrow$ n+1

$$\therefore \quad E = 0 \,, \quad P = 0 \,, \quad m = \frac{L}{2}$$

1 particle ("impurity" or "magnon"):

$$|\psi(p)\rangle = \sum_{x=1}^{L} e^{ipx} |\overset{1}{\overset{\downarrow}{Z}} \cdots \overset{x}{\overset{\downarrow}{X}} \cdots \overset{L}{\overset{\downarrow}{Z}}\rangle$$

$$E = \epsilon(p) = 4 \sin^2 \frac{p}{2} \,, \quad P = p \,, \quad m = \frac{L}{2} - 1$$

2 particles:

"in" "out"

$$|\psi(p_1,p_2)\rangle = A_{XX}(12)|X(p_1)X(p_2)\rangle + A_{XX}(21)|X(p_2)X(p_1)\rangle$$

$$|X(p_i)X(p_j)\rangle \equiv \sum_{x_1<x_2} e^{i(p_i x_1 + p_j x_2)}|\overset{1}{\overset{\downarrow}{Z}}\cdots\overset{x_1}{\overset{\downarrow}{X}}\cdots\overset{x_2}{\overset{\downarrow}{X}}\cdots\overset{L}{\overset{\downarrow}{Z}}\rangle$$

$$H|\psi(p_1,p_2)\rangle = E|\psi(p_1,p_2)\rangle \quad \Rightarrow \quad E = \epsilon(p_1) + \epsilon(p_2)$$

$$A_{XX}(21) = S(p_2\,,p_1)\,A_{XX}(12)$$

S-matrix

$$S(p_2\,,p_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i}$$

$$u_j = u(p_j)\,, \quad u(p) = \frac{1}{2}\cot(\frac{p}{2})$$

Note: For $p_1=p_2\equiv p$ $|\psi(p,p)\rangle = 0$ $\therefore$ p_i must be distinct!

M particles: Introduce ZF operators: [Zamolodchikov[2] 79, Faddeev 80]

$$A^\dagger(p_1)\,A^\dagger(p_2) = S(p_1,p_2)\,A^\dagger(p_2)\,A^\dagger(p_1)$$

$$|\psi\rangle = \sum_{1\leq x_{Q_1}<\ldots<x_{Q_M}\leq L} \Psi^{(Q)}(x_1,\ldots,x_M)|\overset{\overset{1}{\downarrow}}{Z}\cdots\overset{\overset{x_{Q_1}}{\downarrow}}{X}\cdots\overset{\overset{x_{Q_M}}{\downarrow}}{X}\cdots\overset{\overset{L}{\downarrow}}{Z}\rangle$$

In the sector $Q=(Q_1,\ldots,Q_M)$ where $x_{Q_1}<\ldots<x_{Q_M}$

$$\Psi^{(Q)}(x_1,\ldots,x_M) = \sum_P A^P e^{ip_P\cdot x_Q}$$

Sum over all permutations of $P=(P_1,\ldots,P_M)$

$$p_P\cdot x_Q = \sum_{k=1}^N p_{P_k} x_{Q_k}$$

Amplitudes related by $A^P \sim A^\dagger(p_{P_1})\ldots A^\dagger(p_{P_M})$

Example: M=2

In the sector $Q=(1,2)$ where $x_1 < x_2$

$$\Psi^{(12)}(x_1,x_2) = A^{12} e^{i(p_1 x_1 + p_2 x_2)} + A^{21} e^{i(p_2 x_1 + p_1 x_2)}$$

$$A^{21} \sim A^\dagger(p_2) A^\dagger(p_1) = S(p_2,p_1)\, A^\dagger(p_1) A^\dagger(p_2) \sim S(p_2,p_1)\, A^{12}$$

Recover previous results upon identifying

$$A_{XX}(12) = A^{12}\,, \quad A_{XX}(21) = A^{21}$$

key point: multi-particle wave function is constructed using just 2-particle S-matrix!
(Not possible if H is not integrable)

$$E = \sum_{k=1}^{M} \epsilon(p_k)\,, \quad P = \sum_{k=1}^{M} p_k\,, \quad m = \frac{L}{2} - M$$

In terms of u:

$$\epsilon(p) = 4\sin^2\frac{p}{2} = \frac{1}{u^2 + \frac{1}{4}}\,, \quad p = \frac{1}{i}\ln\left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right)$$

same as before!

But where are the Bethe equations?

Come from PBCs, which we have so far neglected!

PBCs:

$$\Psi(1, x_2, \ldots, x_M) = \Psi(L+1, x_2, \ldots, x_M)$$

$$\Rightarrow$$

$$\Psi^{(1\ldots M)}(1, x_2, \ldots, x_M) = \Psi^{(2\ldots M1)}(L+1, x_2, \ldots, x_M)$$

$$A^{1\ldots M} e^{i(p_1 + p_2 x_2 + \ldots + p_M x_M)} + \ldots = A^{2\ldots M1} e^{i(p_1 L + p_1 + p_2 x_2 + \ldots + p_M x_M)} + \ldots$$

$$\Rightarrow$$

$$A^{1\ldots M} = \boxed{A^{2\ldots M1} e^{ip_1 L}}$$

← equal!

But $A^{1\ldots M} \sim A^\dagger(p_1) A^\dagger(p_2) \cdots A^\dagger(p_M)$

$$= [\prod_{j=2}^{M} S(p_1, p_j)] A^\dagger(p_2) \cdots A^\dagger(p_M) A^\dagger(p_1) \sim \boxed{[\prod_{j=2}^{M} S(p_1, p_j)] A^{2\ldots M1}}$$

$\Rightarrow$

$$e^{ip_1 L} = \prod_{j=2}^{M} S(p_1, p_j)$$

Similarly for $\cdots$ terms $\therefore$

$$\boxed{e^{ip_k L} = \prod_{\substack{j=1 \\ j\neq k}}^{M} S(p_k, p_j)\,, \quad k = 1, \ldots, M}$$

"Bethe-Yang equations"

In terms of u:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1\,, \cdots\,, M$$

Our previous BEs!

R-matrix versus S-matrix

	R-matrix	S-matrix
	Has symmetry of Hamiltonian	Has symmetry of vacuum
XXX:	4 x 4 matrix SU(2)-invariant	phase U(1)-invariant
	Gives higher conserved charges; integrability is manifest	

higher-rank generalization

Example: SU(3) subsector

Z, X, Y $\qquad C^3 \otimes \cdots \otimes C^3$

$$H = \sum_{n=1}^{L} (I - \mathcal{P}_{n,n+1})$$

PBCs $L + 1 \equiv 1$

vacuum state: $|0\rangle = |Z \cdots Z\rangle$

1-particle states:

$$|X(p)\rangle = \sum_{x=1}^{L} e^{ipx} |\overset{\overset{1}{\downarrow}}{Z} \cdots \overset{\overset{x}{\downarrow}}{X} \cdots \overset{\overset{L}{\downarrow}}{Z}\rangle$$

$$|Y(p)\rangle = \sum_{x=1}^{L} e^{ipx} |\overset{\overset{1}{\downarrow}}{Z} \cdots \overset{\overset{x}{\downarrow}}{Y} \cdots \overset{\overset{L}{\downarrow}}{Z}\rangle$$

$$E = \epsilon(p) = 4 \sin^2 \frac{p}{2}, \quad P = p$$

2-particle states I:

one X & one Y:

$$|\psi\rangle = A_{XY}(12)|X(p_1)Y(p_2)\rangle + A_{XY}(21)|X(p_2)Y(p_1)\rangle$$

$$+A_{YX}(12)|Y(p_1)X(p_2)\rangle + A_{YX}(21)|Y(p_2)X(p_1)\rangle$$

$$|\phi_1(p_i)\phi_2(p_j)\rangle = \sum_{x_1<x_2} e^{i(p_i x_1 + p_j x_2)}\,|\,\overset{\overset{1}{\downarrow}}{Z}\cdots\overset{\overset{x_1}{\downarrow}}{\phi_1}\cdots\overset{\overset{x_2}{\downarrow}}{\phi_2}\cdots\overset{\overset{L}{\downarrow}}{Z}\rangle$$

$$\begin{pmatrix} A_{XY}(21) \\ A_{YX}(21) \end{pmatrix} = \begin{pmatrix} R(p_2\,,p_1) & T(p_2\,,p_1) \\ T(p_2\,,p_1) & R(p_2\,,p_1) \end{pmatrix} \begin{pmatrix} A_{XY}(12) \\ A_{YX}(12) \end{pmatrix}$$

$$T(p_2\,,p_1) = \frac{u_2 - u_1}{u_2 - u_1 - i}\,, \quad R(p_2\,,p_1) = \frac{i}{u_2 - u_1 - i}$$

2-particle states II:

including also XX & YY states:

$$\begin{pmatrix} A_{XX}(21) \\ A_{XY}(21) \\ A_{YX}(21) \\ A_{YY}(21) \end{pmatrix} = \mathbf{S} \cdot \begin{pmatrix} A_{XX}(12) \\ A_{YX}(12) \\ A_{XY}(12) \\ A_{YY}(12) \end{pmatrix} = \underbrace{\begin{pmatrix} S & & & \\ & T & R & \\ & R & T & \\ & & & S \end{pmatrix}}_{\frac{1}{u_2 - u_1 - i}\left[(u_2 - u_1)I + i\mathcal{P}\right]} \begin{pmatrix} A_{XX}(12) \\ A_{YX}(12) \\ A_{XY}(12) \\ A_{YY}(12) \end{pmatrix}$$

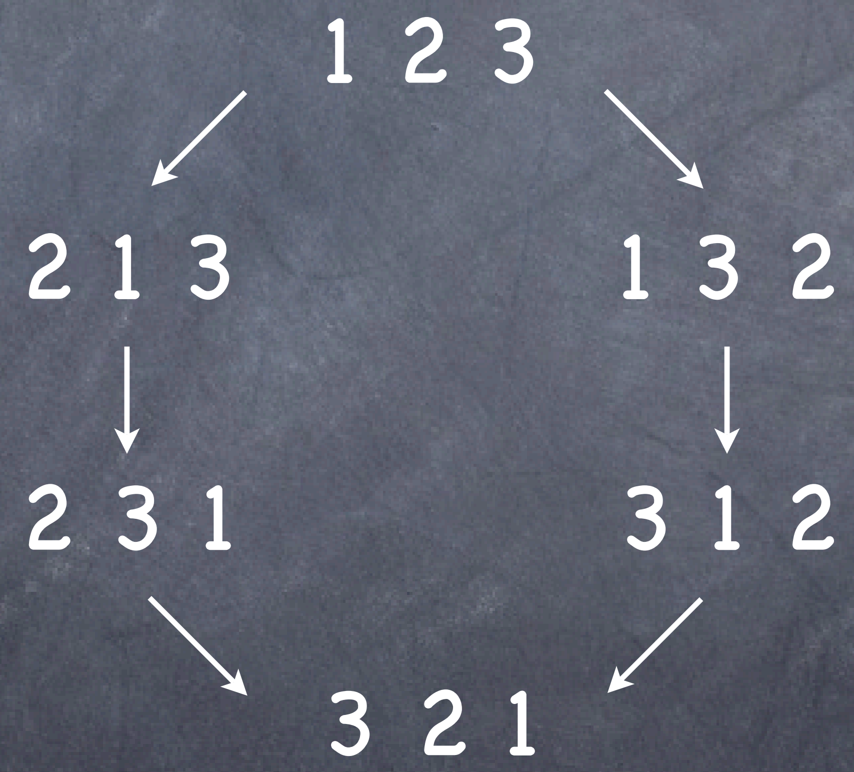
- S-matrix is now really a matrix!
- SU(2)-invariant solution of YBE!

M-particle states: ZF operators now have index!

$$A_i^\dagger(p)\,, \quad i = 1\,,2$$

$$A_i^\dagger(p_1)\, A_j^\dagger(p_2) = S_{i\,j}^{i'\,j'}(p_1,p_2)\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)$$

Associativity for $A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, A_k^\dagger(p_3) \;\Rightarrow$ YBE!



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Associativity for $A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, A_k^\dagger(p_3) \;\Rightarrow$ YBE!

$$|\psi\rangle = \sum_{1 \le x_{Q_1} < \ldots < x_{Q_M} \le L} \; \sum_{i_1, \ldots, i_M = 1}^{2} \Psi^{(Q)}_{i_1 \ldots i_M}(x_1, \ldots, x_M) | \overset{\overset{1}{\downarrow}}{Z} \cdots \overset{\overset{x_{Q_1}}{\downarrow}}{\phi_{i_1}} \cdots \overset{\overset{x_{Q_M}}{\downarrow}}{\phi_{i_M}} \cdots \overset{\overset{L}{\downarrow}}{Z} \rangle$$

$$\Psi^{(Q)}_{i_1 \ldots i_M}(x_1, \ldots, x_M) = \sum_P A^{P|Q}_{i_1 \ldots i_M} e^{i p_P \cdot x_Q}$$

$$A^{P|Q}_{i_1 \ldots i_M} \sim A^\dagger_{i_{Q_1}}(p_{P_1}) \ldots A^\dagger_{i_{Q_M}}(p_{P_M})$$

PBCs:

$$\Psi_{i_1 \ldots i_M}(1, x_2, \ldots, x_M) = \Psi_{i_1 \ldots i_M}(L+1, x_2, \ldots, x_M)$$

Generalizing steps of SU(2) case, [review: Ahn & N 10]

$$\boxed{e^{ip_k L} = -\Lambda(p_k; p_1, \ldots, p_M)\,, \quad k = 1, \ldots, M}$$

Bethe-Yang equations

$\Lambda(p; p_1, \ldots, p_M)$ eigenvalues of (inhomogeneous) transfer matrix

$$t(p; p_1, \ldots, p_M) = \mathrm{tr}_0\, S_{01}(p, p_1) \cdots S_{0M}(p, p_M)$$

But we already determined these eigenvalues using algebraic Bethe ansatz:

$$\Lambda(p;p_1,\ldots,p_M) = \frac{1}{\prod_{l=1}^{M}(u-u_l-i)}\left\{\prod_{l=1}^{M}(u-u_l+i)\prod_{l=1}^{m}\left(\frac{u-\lambda_l-\frac{i}{2}}{u-\lambda_l+\frac{i}{2}}\right) + \prod_{l=1}^{M}(u-u_l)\prod_{l=1}^{m}\left(\frac{u-\lambda_l+\frac{3i}{2}}{u-\lambda_l+\frac{i}{2}}\right)\right\}$$

where

$$\prod_{l=1}^{M}\frac{\lambda_k-u_l+\frac{i}{2}}{\lambda_k-u_l-\frac{i}{2}} = \prod_{\substack{j=1\\ j\neq k}}^{m}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i}$$

$\therefore$

$$e^{ip_kL} = -\Lambda(p_k;p_1,\ldots,p_M) = -\prod_{l=1}^{M}\frac{u_k-u_l+i}{u_k-u_l-i}\prod_{l=1}^{m}\frac{u_k-\lambda_l-\frac{i}{2}}{u_k-\lambda_l+\frac{i}{2}}$$

SU(3) Bethe eqs

Epilogue

- 1-loop dilatation operator for all single-trace operators in $\mathcal{N}$=4 SYM is integrable

$\Rightarrow$

- 1-loop anomalous dimensions are given by a set of Bethe equations

Higher loops?

Next time:
- all-loop S-matrix!
- all-loop Bethe equations!