Introduction to Gauge/Gravity Duality

Lecture 2

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Abstract

A series of four lectures given at the Australian National University in December 2011 for the 2nd Asia-Pacific Summer School in Mathematical Physics/22nd Canberra International Physics Summer School.

1 Anti-de Sitter Space

The geometry of AdS space plays a crucial role in many aspects of holography[1] although there is some work on holography for more general backgrounds [3, 4]. We will more or less discuss AdS_5 but most features of AdS space are dimension independant.

Algebraically, AdS_5 space is given by the hypersurface

$$L^2 = -X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 \tag{1}$$

for some constant R. The metric on AdS_5 is induced from the flat metric of signature $(2, 4)$

$$ds_{2,4}^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2. \tag{2}$$

Perhaps the two timelike directions of the ambient cause alarm but one should reassure oneself that on the hypersurface given by (1) there is just one timelike direction.

[1] Two references for details regarding AdS space are [1] and [2]

1.1 Global AdS

These cartestian co-ordinates are an inconvenient parameterization of AdS_5, one more natural choice of co-ordinates is

$$\begin{aligned} X_0 &= L\cosh\rho\cos\tau\,, & (3)\\ X_1 &= L\cosh\rho\sin\tau\,, & (4)\\ X_{1+i} &= L\sinh\rho\,\Omega_i\,,\quad i=1,\ldots,4\,, & (5)\end{aligned}$$

where Ω_i parameterize three dimensional sphere of unit radius:

$$\sum_{i=1}^{4}\Omega_i = 1\,. \tag{6}$$

We then find the metric on AdS_5 to be

$$ds^2_{AdS_5} = L^2\big(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2\big) \tag{7}$$

where $d\Omega_3^2$ is the line element on the unit three-sphere.

Exercise: Show that the above metric for AdS_5 solves Einstein's equation with a cosmological constant

$$S_{R,\Lambda} = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\Big(R-\Lambda\Big) \tag{8}$$

and relate the cosmological constant to the radius L of AdS_5.

It is instructive to understand better the geometry and topology of AdS_5. The co-ordinate τ is timelike and has period 2π which is essentially the defining feature of closed timelike curves. These are eliminated by working on the universal covering space[2]. Further, we see that as $\rho\to\pm\infty$ the radius of the τ-circle grows without bound while at $\rho=1$ is has radius R. As such we can visualize AdS_5 as a doubly-trumpeted cylinder with an S^3 at each point. This S^3 smoothly shrinks to zero size at $\rho=1$.

The metric (7) preserves all the symmetries of the hyperboloid equation (1) namely

$$G = SO(4,2)\,, \tag{9}$$

in fact AdS_5 is a coset space

$$AdS_5 = \frac{SO(4,2)}{SO(4,1)}\,. \tag{10}$$

As a quick check we see that $\dim SO(4,2)=15$ and $\dim SO(4,1)=10$.

1.2 The Boundary of AdS Space

Perhaps the most crucial aspect of gauge/gravity duality is that the space-time on which the gravity theory lives has a non-trivial boundary on which the QFT is defined. As such,

[2]The universal covering space of a manifold M is obtained by chosing an arbitrary basepoint x_0 and considering the space of all paths in M originating at x_0, up to continuous deformations.

it behooves us to understand the boundary of AdS space in some detail. Roughly speaking, the boundary of AdS_5 is $\mathbb{R}_{1,3}$ but one must be a bit more careful and this requires us understanding how to compactify a non-compact manifold.

Perhaps counterintuitively, one compactifies a space by *adding* points, not subtracting points. For example, one can compactify flat Euclidean $\mathbb{R}^n$ to S^n by adding the point at infinity. However the compactification of $\mathbb{R}^{1,p}$ which we will use is a bit more subtle than this.

First we take a small detour, we consider the space of null rays in $\mathbb{R}^{2,p}$. Denote the flat signature $(2,p)$ metric by η_{ab} then null rays are of course the space of lines which satisfy

$$\eta_{ab}X^a X^b = 0. \tag{11}$$

This quadric has a canonical action of $SO(2,p)$ (the Lorentz group of $\mathbb{R}^{2,p}$) on it, as well as an action of $\mathbb{R}_+$ by common rescaling $X^a \to \lambda X^a$. We can parameterize this space by $X^\mu \in \mathbb{R}^{1,p}$

$$\widetilde{\mathbb{R}}^{1,p} = (\frac{1}{2}(1+X^2), X^\mu, \frac{1}{2}(1-X^2))\,, \tag{12}$$

where the tilde represents that in fact this is a compactification of $\mathbb{R}^{1,p}$.

Now we show that this space just described is the boundary of AdS_{p+2}. We start with the definition of AdS_{p+2}

$$\eta_{ab}X^a X^b = L^2 \tag{13}$$

then under $X_a = \lambda \widetilde{X}_a$ we get the same AdS space in new coordinates

$$\eta_{ab}\widetilde{X}^a \widetilde{X}^b = \frac{L^2}{\lambda^2}\,. \tag{14}$$

Now to describe the boundary we take the limit $\lambda \to \infty$ and mod out by constant rescalings:

$$\widetilde{\mathbb{R}}^{1,p} = \frac{\{\widetilde{X}^a \,|\, \eta_{ab}\widetilde{X}^a\widetilde{X}^b = 0\}}{\{\widetilde{X}^a \sim c\widetilde{X}^a\}}\,. \tag{15}$$

Now the punchline which really gives the game away will have to wait until next lectures when we learn about conformal transformations but perhaps some people have been studying independantly ;)...The Lorentz group of $\mathbb{R}^{2,p}$ is the conformal group of metrics on $\mathbb{R}^{1,p}$. This group acts in a canonical way on all interior point of AdS_p since it preserves the quadratic form $\eta_{ab}X^a X^b$ but it acts on the boundary of AdS in a pretty non-trivial way. In terms of X^μ in (12) it acts as *conformal transformations*.

1.3 The Poincaré Patch

We can solve (1) in a quite different manner:

$$X_0 = \frac{1}{2u}\big(1 + u^2(L^2 + \vec{x}^2 - t^2)\big)\,, \tag{16}$$

$$X_1 = Lut\,, \tag{17}$$

$$X_{1+i} = Lux^i\,, \quad i = 1,2,3\,, \tag{18}$$

$$X_5 = \frac{1}{2u}\big(1 + u^2(L^2 - \vec{x}^2 + t^2)\big)\,. \tag{19}$$

This coordinate system does not cover all of AdS space, it is known as the Poincaré patch. That is not a problem. The metric in these co-ordinates is

$$ds^2 = L^2\Big(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\Big) \tag{20}$$

One can see from (7) and (20) we see that in global AdS, τ is a global time coordinate whereas in the Poincaré patch, there is a horizon at $u = 0$ where the time coordinate t becomes null.

Another feature of the Poincaré patch is that the full symmetry group is not manifest since some generators would take a point on the manifold outside the patch. What is manifest is just

$$G_{\text{Poincare}} = SO(1,3) \times \mathbb{R}^4 \times SO(1,1) \tag{21}$$

where the $SO(1,1)$ factor is dilatations

$$(t, \vec{x}, u) \to (\lambda t, \lambda\vec{x}, \lambda^{-1}u)\,. \tag{22}$$

2 Scalar Fields in Anti-de Sitter Space

The gauge gravity duality relates fluctuations in the bulk to the physics of the boundary QFT so it is of primary interest to understand exactly how fields propagate in the bulk.

We will work in Euclidean AdS

Consider a massive scalar field with action

$$\mathcal{S} = -\frac{1}{2}\int d^{d+1}x\sqrt{g}\Big(\nabla_\mu\phi\nabla^\mu\phi + m^2\phi^2\Big) \tag{23}$$

with equation of motion

$$\Box\phi = m^2\phi. \tag{24}$$

Recalling the formula

$$\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu\big(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\big) \tag{25}$$

and using the metric

$$ds^2_{AdS_{d+1}} = \frac{1}{z^2}\Big(ds^2_d + dz^2\Big) \tag{26}$$

with the ansatz

$$\phi = e^{ik\cdot x}z^{(d-1)/2}\psi(z) \tag{27}$$

one can show that (24) becomes

$$\psi''(z) + \Big(\frac{-m^2 + (1-d^2)/4}{z^2}\Big)\psi(z) - k^2\psi = 0\,. \tag{28}$$

We are interested in zero momentum solutions which are given in terms of Bessel functions

$$\psi(z) = z^{1/2}\Big[c_1 I_\lambda(|k|z) + c_2 K_\lambda(|k|z)\Big] \tag{29}$$

with

$$\lambda = \sqrt{m^2 + \frac{d^2}{4}}\,. \tag{30}$$

We would like to allow such solutions when the energy is positive and disallow them otherwise. Due to the curvature of AdS space the potential is singular in this Schrodinger equation (28) and the issue of positivity of the energy is a moderately subtle affair. The key issue is the boundary condition at $z = 0$, one must enforce that no energy leaks out of AdS space. The final analysis is that mildly negative mass states are allowed

$$m^2 \geq -\frac{d^2}{4}\,. \tag{31}$$

This fundamental result is known as the Breitenlohner-Freedman bound [5]

2.1 Geodesics

References

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