# Lecture 2. Classical string integrability

# Green-Schwarz action on $AdS_5$ x $S^5$

$$S = \frac{1}{\alpha'} \int d^2\xi \Big[ \sqrt{-g} g^{ab} G_{\mu\nu}(x) \partial_a x^\mu \partial_b x^\nu + ( \sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ} ) \bar{\theta}^I \rho_a D_b \theta^J + O(\theta^4) \Big]$$

- 2d worldsheet coordinates $\xi^a = (\tau, \sigma)$ and metric $g^{ab}$
- Majorana-Weyl spinor fields
- projections of 10-d Dirac matrices
- projections of 10-d covariant derivatives

# Metsaev-Tseytlin action (1998)

- Sigma model on super-coset manifold

$$AdS_5 \times S^5 \approx \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)} \quad \rightarrow \quad \frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$$

- $Z_4$ grading
  - Decomposition of $g \in \mathrm{psu}(2,2|4) \rightarrow g^{(0)} \oplus g^{(1)} \oplus g^{(2)} \oplus g^{(3)}$
  - $g^{(0)} = h \in \mathrm{so}(4,1) \times \mathrm{so}(5)$
  - Commutation relations $[g^{(m)}, g^{(n)}] \subset g^{(p)}, \quad p = m + n \bmod 4$

- String action
  - Left-invariant current $j_a^{(n)} = -g^{(n)^{-1}} \partial_a g^{(n)}$
  - Nonlinear sigma-model with WZW term:

$$S = \frac{R^2}{\alpha'} \int d^2\xi \, \mathrm{STr}\left[g^{ab} j_a^{(2)} j_b^{(2)} + \epsilon^{ab} j_a^{(1)} j_b^{(3)}\right]$$

# Bena-Polchinski-Roiban (2003)

- Lax pair construction (cf) Lecture by V. Bazhanov
  - Lax pair :

$$L_a(x) = j_a^{(0)} + \frac{x^2+1}{x^2-1} j_a^{(2)} - \frac{2x}{x^2-1} \epsilon_{ab} j_b^{(2)} + \sqrt{\frac{x+1}{x-1}} j_a^{(1)} + \sqrt{\frac{x-1}{x+1}} j_a^{(3)}$$

  - Zero curvature condition : $\partial_a L_b - \partial_b L_a - [L_a, L_b] = 0$
  - Monodromy matrix : $\Omega(x) = \mathcal{P} \exp \int_\gamma L_a(x) d\xi^a$
  - Infinite numbers of classically conserved charges

$$T(x) = \mathrm{STr}\, \vec{\exp} \int_0^{2\pi} L_\sigma(x) d\sigma$$

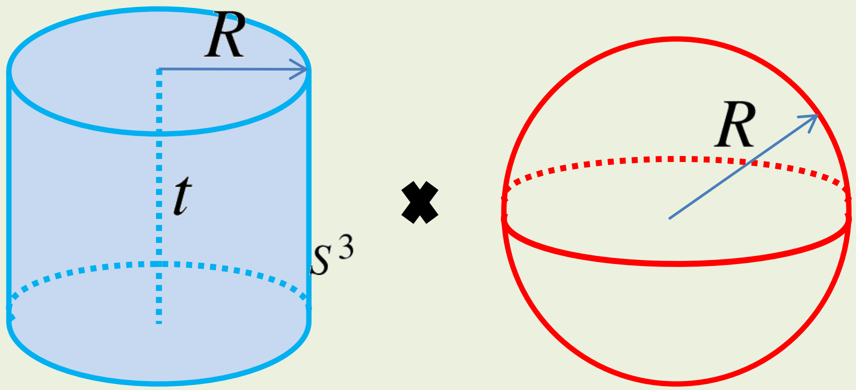
# Two approaches on classical string theory

- (cf) V. Bazhanov's yesterday lectures
  - Explicitly solve KdV equation: "soliton"
  - Use integrability structure: Lax pair
- Explicit classical string solutions
  - Directly compute energies of string configurations
  - Fermions decoupled
  - Difficult to relate with gauge theory side
- Algebraic curves based on the integrability
  - No direct link to classical string configurations
  - Quantum strings can be included easily
  - Can relate to strong coupling limit of all-loop Bethe ansatz eqs.

# Classical bosonic strings on $AdS_5 \times S^5$

- $AdS_5$ and $S^5$ are decoupled in bosonic action

$$S = \frac{R^2}{\alpha'} \int d^2\xi \left[ G^{(S^5)}_{mn} \partial_a X^m \partial^a X^n + G^{(AdS)}_{mn} \partial_a Y^m \partial^a Y^n \right]$$




- $AdS_5$ and $S^5$ embedding coordinates

$$X_1^2 + \cdots + X_6^2 = 1, \qquad Y_0^2 - Y_1^2 - \cdots - Y_4^2 + Y_5^2 = 1$$

- Virasoro constraints

$$\dot{X}^m X'_m + \dot{Y}^n Y'_n = 0, \quad \dot{X}^m \dot{X}_m + \dot{Y}^n \dot{Y}_n + X'^m X'_m + Y'^n Y'_n = 0$$

- String action
$$S = \frac{R^2}{\alpha'} \int d^2\xi \left[ -\partial_a X^m \partial^a X_m + \Lambda(X^2 - 1) - \partial_a Y^m \partial^a Y_n + \tilde{\Lambda}(Y^2 + 1) \right]$$

- Eqs. of motion
$$\begin{aligned} \partial^a \partial_a Y_n - \tilde{\Lambda} Y_n &= 0, \qquad \tilde{\Lambda} = \partial^a Y_n \partial_a Y^n, \qquad Y_n Y^n = -1 \\ \partial^a \partial_a X_m - \Lambda X_m &= 0, \qquad \Lambda = \partial^a X_m \partial_a X^m, \qquad X_m X^m = 1 \end{aligned}$$

- Conserved charges
$$S_{pq} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (Y_p \dot{Y}_q - Y_q \dot{Y}_q), \quad J_{mn} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (X_m \dot{X}_n - X_n \dot{X}_m)$$
$$(S_{50}, S_{12}, S_{34} | J_{12}, J_{34}, J_{56}) \quad \leftrightarrow \quad (\Delta, S_1, S_2 | J_1, J_2, J_3)$$

- Global coordinates
$$\begin{aligned} &Y_1 + iY_2 = \sinh\rho \cos\psi e^{i\phi_1}, \qquad Y_3 + iY_4 = \sinh\rho \sin\psi e^{i\phi_2}, \\ &Y_5 + iY_0 = \cosh\rho e^{it}, \qquad X_5 + iX_6 = \cos\gamma e^{i\varphi_3}, \\ &X_1 + iX_2 = \sin\gamma \cos\theta e^{i\varphi_1}, \qquad X_3 + iX_4 = \sin\gamma \sin\theta e^{i\varphi_2} \end{aligned}$$
$$\begin{aligned} (ds^2)_{AdS_5} &= R^2 \left[ d\rho^2 - \cosh^2\rho dt^2 + \sinh^2\rho (d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2) \right] \\ (ds^2)_{S^5} &= R^2 \left[ d\gamma^2 + \cos^2\gamma d\varphi_3^2 + \sin^2\gamma (d\theta^2 + \cos^2\theta d\varphi_1^2 + \sin^2\theta d\varphi_2^2) \right] \end{aligned}$$

- 6 isometry coordinates $\quad t,\ \phi_1,\ \phi_2,\ \varphi_1,\ \varphi_2,\ \varphi_3$

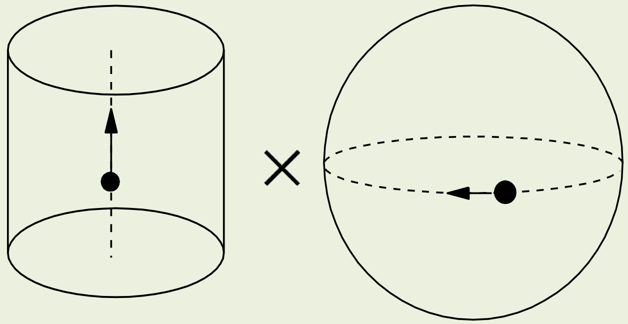
# BPS string

- A point-like [no σ dependence] string which rotates on a great circle of $S^5$ with angular momentum $J \to \infty$

$$Y_5 + iY_0 = e^{i\kappa\tau},\ X_1 + iX_2 = e^{i\kappa\tau},\ \kappa = \sqrt{\Lambda},\ Y_{1,2,3,4} = X_{3,4,5,6} = 0$$

- Energy = angular momentum

$$E = J_1 = \sqrt{\lambda}\kappa$$



# BMN string

Berenstein, Maldacena, Nastase (2002)

- A point-like string in the planar-limit $R \to \infty$ with $\rho,\ \gamma \to 0$

$$
\begin{aligned}
(ds^2)_{AdS_5} &= R^2\left[d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho \underbrace{\boxed{(d\psi^2 + \cos^2\psi\, d\phi_1^2 + \sin^2\psi\, d\phi_2^2)}}_{d\Omega_3^2}\right] \\
(ds^2)_{S^5} &= R^2\left[d\gamma^2 + \cos^2\gamma\, d\varphi_3^2 + \sin^2\gamma \underbrace{\boxed{(d\theta^2 + \cos^2\theta\, d\varphi_1^2 + \sin^2\theta\, d\varphi_2^2)}}_{d\Omega_3'^2}\right]
\end{aligned}
$$

$$
ds^2 = R^2\left[d\rho^2 - (1+\rho^2)dt^2 + d\gamma^2 + (1-\gamma^2)d\varphi_3^2 + \rho^2(d\Omega_3)^2 + \gamma^2(d\Omega_3')^2\right] + O(R^{-2})
$$

$$
(r \equiv R\rho,\ \ y \equiv R\gamma)
$$

$$
= R^2(d\varphi_3^2 - dt^2) + dr^2 - r^2dt^2 + dy^2 - y^2 d\varphi_3^2 + r^2(d\Omega_3)^2 + y^2(d\Omega_3')^2 + O(R^{-2})
$$

$$
t, \varphi_3 = \mu x^+ \pm \frac{x^-}{\mu R^2}
$$

$$
= -4dx^+dx^- - \mu^2\underbrace{(r^2 + y^2)}_{\vec{z}^2 + \vec{y}^2 \equiv \vec{x}^2}(dx^+)^2 + \underbrace{dr^2 + dy^2 + r^2(d\Omega_3)^2 + y^2(d\Omega_3')^2}_{d\vec{z}^2 + d\vec{y}^2 \equiv d\vec{x}^2}
$$

- Light-cone gauge: $x^+ = \tau$
- Orthogonal directions have only quadratic fluctuations

$$S = \frac{1}{\alpha'}\int d\tau \int_0^{\alpha' p^+} d\sigma \left[\frac{1}{2}(\partial_a \vec{x})^2 - \frac{\mu^2}{2}\vec{x}^2\right] \quad \text{with} \quad p^+ = \frac{J}{\mu R^2}$$

- Light cone energy exact in all orders of $\lambda$

$$2p^- = E - J = \sum_{n=-\infty}^{\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} = \mu \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \lambda \frac{n^2}{J^2}}$$

- Relation to the SYM $(\mu \equiv 1)$

$$|0\rangle \quad \leftrightarrow \quad \mathrm{Tr}\left[Z^J\right], \qquad E = J$$

$$a^i_{-n} a^i_n |0\rangle \quad \leftrightarrow \quad \mathrm{Tr}\left[X^2 Z^J + \ldots\right], \qquad E = J + 2\sqrt{1 + \lambda \frac{n^2}{J^2}}$$

# Neumann-Rosochatius reduction

Arutyunov, Russo, Tseytlin (2002)

- A string in $R_t \times S^3 \;\rightarrow\; \rho = 0,\; \gamma = \frac{\pi}{2},\; \varphi_3 = 0$

$$t = \kappa\tau,\; \cos\theta(\sigma,\tau) = r_1(\chi),\; \sin\theta(\sigma,\tau) = r_2(\chi),\; \varphi_j(\sigma,\tau) = \omega_j\tau + f_j(\chi),\; \chi = \alpha\sigma + \beta\tau$$

- Effective 1d integrable Lagrangian (Neumann-Rosochatius)

$$L_{NR} = (\alpha^2 - \beta^2)\sum_{j=1}^{2}\left[r_j'^2 - \frac{1}{(\alpha^2-\beta^2)^2}\left(\frac{C_j^2}{r_j^2} + \alpha^2\omega_j^2 r_j^2\right)\right] + \Lambda\left(\sum_{j=1}^{2} r_j^2 - 1\right)$$

- Conserved charges and Virasoro constraints

$$E = \frac{\lambda}{2\pi}\frac{\kappa}{\alpha}\int d\chi, \quad J_j = \frac{\lambda}{2\pi}\frac{1}{\alpha^2-\beta^2}\int d\chi\left(\frac{\beta}{\alpha}C_j + \alpha\omega_j r_j^2\right), \qquad \sum_{j=1}^{2} C_j\omega_j + \beta\kappa^2 = 0$$

- Exact solution $E=\frac{\sqrt{\lambda}}{2\pi}\mathcal{E},\quad J=\frac{\sqrt{\lambda}}{2\pi}\mathcal{J},\quad v=-\frac{\beta}{\alpha}$

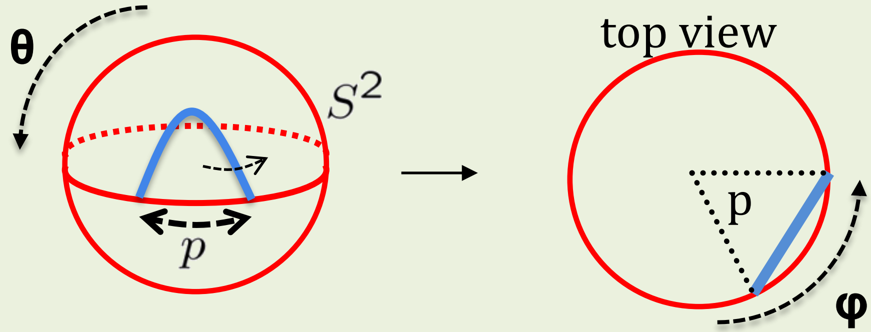
$$\begin{aligned}
\mathcal{E} &= 2\sqrt{(1-v^2)(1-\epsilon)}\mathbf{K}(1-\epsilon),\quad \mathcal{J}=2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{K}(1-\epsilon)-\mathbf{E}(1-\epsilon)\right],\\
\mathcal{E}-\mathcal{J} &= 2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{E}(1-\epsilon)-\left(1-\sqrt{(1-v^2\epsilon)(1-\epsilon)}\right)\mathbf{K}(1-\epsilon)\right],\\
p &= 2v\sqrt{\frac{1-v^2\epsilon}{1-v^2}}\left[\frac{1}{v^2}\Pi\left(1-\frac{1}{v^2}|1-\epsilon\right)-\mathbf{K}(1-\epsilon)\right]
\end{aligned}$$

- Infinite J limit

$$\underbrace{\mathcal{E}-\mathcal{J}=2\sin(p/2)}_{\text{Giant magnon}}\left[1\overset{\text{Finite-size correction}}{\boxed{-4\sin^2(p/2)\exp\left(-\frac{\mathcal{J}}{\sin(p/2)}-2\right)}}\right]$$

# Giant magnon

- Classical string configuration in R x $S^2$ **Hofman, Maldacena (2006)**




$$\cos\theta = \frac{\sin\frac{p}{2}}{\cosh\xi}, \qquad \tan\varphi = \tan\frac{p}{2}\tanh\xi, \qquad \xi \equiv \frac{\sigma - \cos\frac{p}{2}\tau}{\sin\frac{p}{2}}$$

  - Energy of the string $\qquad E = \frac{\sqrt{\lambda}}{\pi}\sin\frac{p}{2}$

  - Pohlmeyer reduction: $S^2$ angle θ is related to the sine-Gordon field, GM is mapped to the SG "Soliton" $\quad \Phi = 2\tan^{-1}(e^{\xi})$

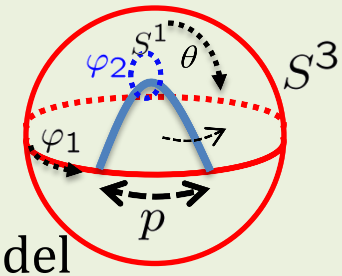
  - Dual to magnons in the SYM spin chain $\qquad \cdots \Uparrow\Uparrow \Downarrow \Uparrow\Uparrow \cdots$

# Dyonic giant magnon

Chen, Dorey, Okamura (2007)

- GM in $R\times S^3 \to |Z_1|^2+|Z_2|^2=1 \ (Z_1=\sin\theta e^{i(\tau+\varphi_1)}, \quad Z_2=\cos\theta e^{i(\omega\tau+\varphi_2)})$

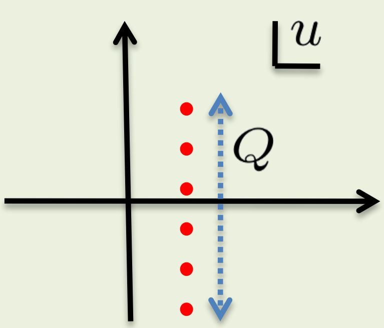
$$\cos\theta=\frac{\sin\frac{p}{2}}{\cosh\tilde{\xi}}, \quad \tan\varphi_1=\tan\frac{p}{2}\tanh\tilde{\xi}, \quad \tilde{\xi}\equiv\alpha\sigma+\beta\tau$$




  – Related to classically integrable complex sine-Gordon model

  – Energy-charge relation: $E-J_1=\sqrt{Q^2+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}, \quad J_2=Q$

  – Dual to magnon bound states "Bethe string"



# Algebraic curves of AdS/CFT

- Lax pair : $$L_a(x) = j_a^{(0)} + \frac{x^2+1}{x^2-1} j_a^{(2)} - \frac{2x}{x^2-1}\epsilon_{ab} j_b^{(2)} + \sqrt{\frac{x+1}{x-1}}\, j_a^{(1)} + \sqrt{\frac{x-1}{x+1}}\, j_a^{(3)}$$
  - Satisfy $\mathrm{STr}[L_a] = 0$

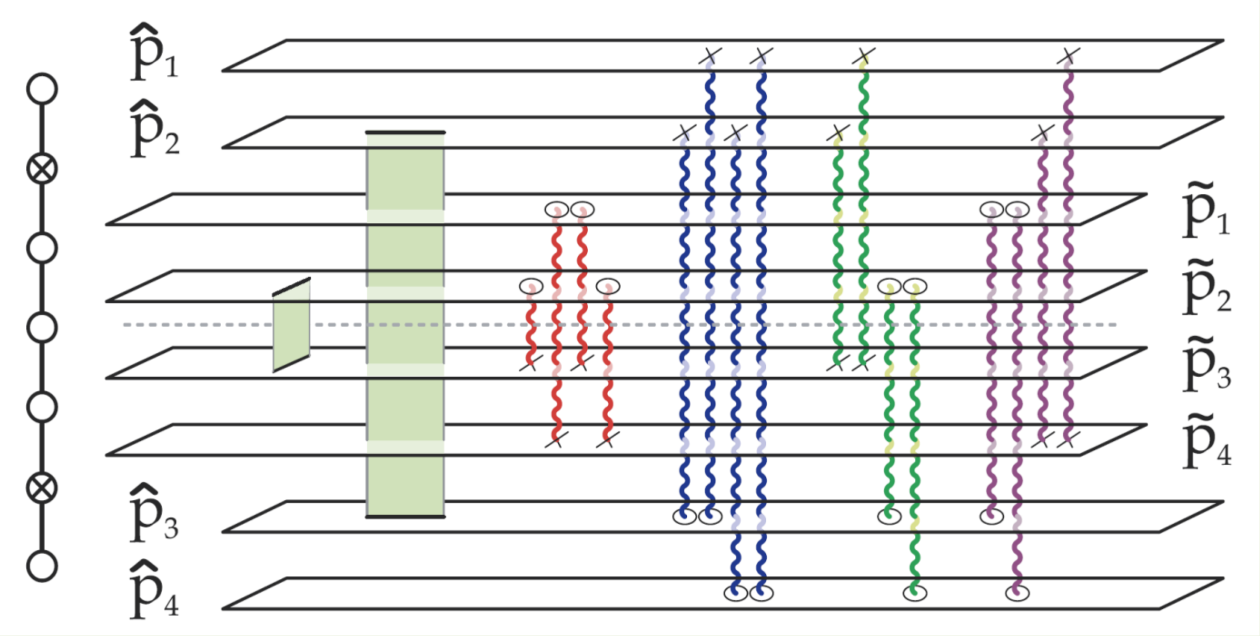
- Monodromy matrix : $$\Omega(x) = \mathcal{P}\exp \oint_\gamma L_a(x) d\xi^a$$
  - Satisfy $\mathrm{SDet}[\Omega(x)] = 1$
  - Eight eigenvalues of the monodromy matrix define quasi-momenta
    $$\Omega(x) \sim \left(e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4} | e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4}\right)$$
  - Unimodularity: $$\sum_{i=1}^{4} [\hat{p}_i(x) - \tilde{p}_i(x)] = 2\pi k, \quad k = \mathbb{Z}$$

- Algebraic curves on eight Riemann sheets
  - Discontinuities across the branch cuts $(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4 | \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$

$$p_i(x+i0) - p_j(x-i0) = 2\pi n_{ij}\,, \qquad x \in \mathcal{C}_n^{ij}$$

$$i \in \{\tilde{1}, \tilde{2}, \hat{1}, \hat{2}\}\,, \qquad j \in \{\tilde{3}, \tilde{4}, \hat{3}, \hat{4}\}$$

$$\begin{aligned} S^5 : &\quad (\tilde{1},\tilde{3})\,,(\tilde{1},\tilde{4})\,,(\tilde{2},\tilde{3})\,,(\tilde{2},\tilde{4}) \\ AdS_5 : &\quad (\hat{1},\hat{3})\,,(\hat{1},\hat{4})\,,(\hat{2},\hat{3})\,,(\hat{2},\hat{4}) \\ \text{Fermions} : &\quad (\tilde{1},\hat{3})\,,(\tilde{1},\hat{4})\,,(\tilde{2},\hat{3})\,,(\tilde{2},\hat{4}) \\ &\quad (\hat{1},\tilde{3})\,,(\hat{1},\tilde{4})\,,(\hat{2},\tilde{3})\,,(\hat{2},\tilde{4})\,. \end{aligned}$$



- Properties of quasi-momenta
  - Virasoro constraint :

$$\{\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4|\tilde{p}_1,\tilde{p}_2,\tilde{p}_3,\tilde{p}_4\} = \frac{\{\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm|\alpha_\pm,\alpha_\pm,\beta_\pm,\beta_\pm\}}{x\pm 1} + \mathcal{O}(1)$$

  - Conserved charges

$$\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hline \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{pmatrix} = \frac{2\pi}{x} \begin{pmatrix} +\mathcal{E} - \mathcal{S}_1 + \mathcal{S}_2 \\ +\mathcal{E} + \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 \\ -\mathcal{E} + \mathcal{S}_1 + \mathcal{S}_2 \\ \hline +\mathcal{J}_1 + \mathcal{J}_2 - \mathcal{J}_3 \\ +\mathcal{J}_1 - \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 \\ -\mathcal{J}_1 - \mathcal{J}_2 - \mathcal{J}_3 \end{pmatrix} + \mathcal{O}\left(\frac{1}{x^2}\right)$$

$$E = \frac{\sqrt{\lambda}}{4\pi} \lim_{x\to\infty} x(\hat{p}_1(x) + \hat{p}_2(x))$$

  - Inversion relation from automorphism of psu(2,2|4)

$$\tilde{p}_{1,2}(x) = -\tilde{p}_{2,1}(1/x) - 2\pi m$$
$$\tilde{p}_{3,4}(x) = -\tilde{p}_{4,3}(1/x) + 2\pi m$$
$$\hat{p}_{1,2,3,4}(x) = -\hat{p}_{2,1,4,3}(1/x)$$

  - Filling fraction

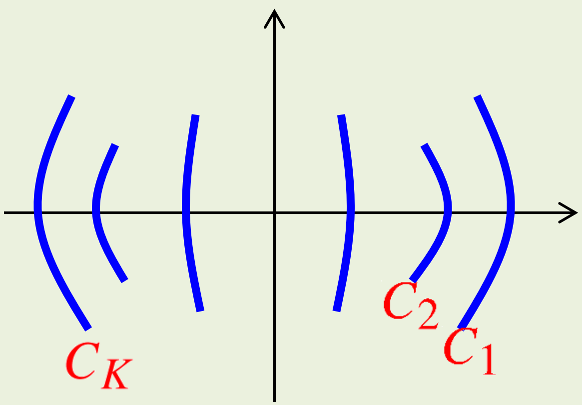
$$S_{ij} = \pm \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{\mathcal{C}_{ij}} \left(1 - \frac{1}{x^2}\right) p_i(x)\,dx$$

# Continuum limit of BAE

- Take limits $L,\ M \to \infty, \quad u_j \sim L$

$$L \ln \frac{u_j + i/2}{u_j - i/2} = \sum_{k \neq j}^{M} \ln \frac{u_j - u_k + i}{u_j - u_k - i} - 2\pi i n_j \quad \xrightarrow[x_j \equiv \frac{u_j}{L}]{} \quad \frac{1}{x_j} = \frac{2}{L} \sum_{k \neq j}^{M} \frac{1}{x_j - x_k} - 2\pi n_j$$

- Bethe roots condensate and form cuts on the complex plane




- Define density of roots $\quad \rho(x) = \frac{1}{L} \sum_{j-1}^{M} \delta(x - x_j)$

- Energy :

$$\gamma = \frac{\lambda}{8\pi^2 L^2} \sum_{j=1}^{M} \frac{1}{x_j^2} = \frac{\lambda}{8\pi^2 L} \int_C \frac{\rho(x)}{x^2} dx$$

- Continuum BAE : $\frac{1}{x} = 2 \int_C dy \frac{\rho(y)}{x - y \pm i0} \pm 2\pi i \rho(x) - 2\pi n_x$

- Resolvent :

$$G(x) = \int_C dy \frac{\rho(y)}{x-y} = \frac{1}{L} \sum \frac{1}{x - x_k}$$

$$G(x+i0) + G(x-i0) = \frac{1}{x} + 2\pi n_x$$

- Quasi-momentum : $p(x) \equiv G(x) - \frac{1}{2x} \rightarrow p(x+i0) + p(x-i0) = 2\pi n_x$

- To complete the match, one needs all-loop and all-sector BAE

# Summary

- Green-Schwarz superstring on AdS5 x S5
- Nonlinear sigma model on coset
- Classical integrability through Lax pair
- Exact solutions such as BPS, BMN, Giant magnons
- Quasi-momenta and algebraic curve through monodromy matrix
- Next: exact worldsheet S-matrix