Quantum chromodynamics

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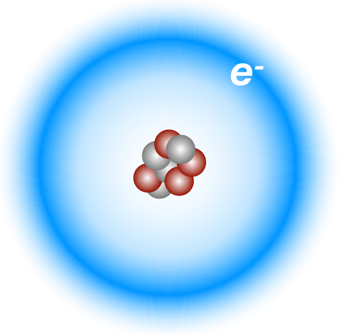
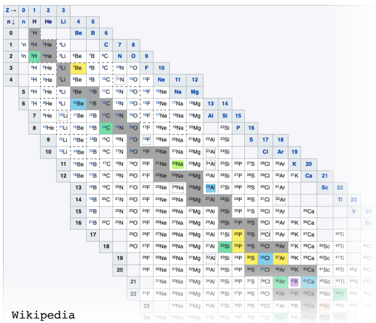
1 Foundations

1.1 What is quantum chromodynamics?



> Quantum chromodynamics (QCD) is the fundamental theory underlying the strong nuclear force — a gauge theory describing the interactions of quarks and gluons.

- Nuclear physics was born with the discovery of the atomic nucleus by Rutherford in 1911.
- Subsequently, Chadwick discoverd the neutron in 1932 — similar in mass to a proton, but electrically neutral.
- 2 basic building blocks to build up the chart of nuclides: protons & neutrons, or collectively "nucleons".



- The forces that bind nuclei must be **strong**:
  - Nuclei are compact.
  - Binding must be sufficient to overcome Coulomb repulsion between protons.
- Yukawa (1935) proposed a (spinless) meson as a potential force carrier with mass scale:

$$m_{\text{pion}} \simeq 200\, m_{\text{electron}} \qquad \left[\simeq \frac{1}{10}\, m_{\text{proton}}\right].$$

  The mass scale was predicted on the basis of the size of nuclei, where massive particle exchange gives rise to potential energy:

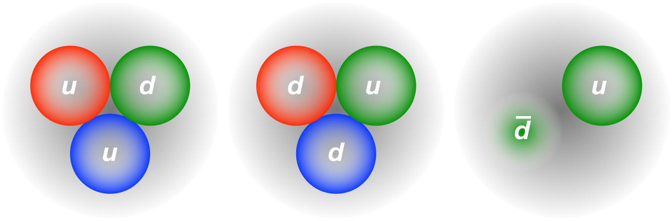
$$V(r) \propto \frac{e^{-mr}}{r}, \qquad \text{“Yukawa potential”}.$$

The class of particles that participate in the strong interaction are collectively referred to as **“hadrons”**. Our simplest (empirical) examples are the proton, neutron and pion... we'll get to talking about a range of other states soon.

- We all know the ultimate solution to this puzzle: *a gauge field theory describing coloured quarks*.
  The **building blocks** of QCD:
  - A set of different quark flavours: “up”, “down”, “strange”...
    Each quark field carries 3 colour charges
  - Interactions of the theory are constructed by demanding a (local) gauge symmetry in this colour degree of freedom.
  - In analogy with the photon of QED, we must have gauge bosons that communicate the colour force: **gluons**.

As we will see, gluons themselves carry colour charge and are therefore are able to directly interact with themselves — a key feature that distinguishes a non-Abelian gauge theory from QED.

- Hadrons are emergent phenomena:



Proton $m_p \simeq 938\,\text{MeV}$ **Neutron** $m_n \simeq 939\,\text{MeV}$ **Pion** $m_\pi \simeq 140\,\text{MeV}$

and a host of other states that we'll come back to...

- It is worth noting that, in the early days, there was a tremendous conceptual challenge to consider the construction of a Lagrangian-based theory that does not contain the observed states.
So how did we get over this conceptual challenge?

1.2 How come QCD?

Here we wish to highlight three key features supported the establishment of QCD: confinement, asymptotic freedom and evidence for the gluon.

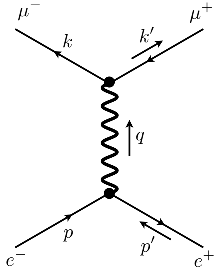
Confinement: the charm revolution.

> The discovery of the charm quark helped to both establish the quark model, and gave support for there being *three* colours of quarks.

- The production of hadrons in electron–positron annihilation gives a relatively clean production process.

> One can also produce hadrons in hadron–hadron collisions, but of course we need then to understand the complexities of both the initial and final states of the process.

- First, let's consider the simpler process of muon production at lowest order in QED



In the relativistic limit, this cross section is given by:

There's nothing too special about the relativistic limit, but it makes the expressions simpler, where for $E_{\text{CM}} \gg 2m_\mu$ the mass terms become negligible.

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi}{3}\frac{\alpha^2}{E_{\text{CM}}^2}, \quad \text{with } \alpha = \frac{e^2}{4\pi}. \tag{1.1}$$

Note of course that σ is frame independent, and we could of course replace E_{CM}^2 with the total invariant mass (squared) of the e^+e^- (or $\mu^+\mu^-$) pair, e.g. s or q^2.

- Now suppose that we have electromagnetically-charged quarks. We'll ignore any possible interactions between the quarks, and just assume that ultimately they will turn into hadrons.

- The calculation takes the same for as the muon case, with the $\gamma\bar{q}q$ vertex modified to include the quark charge factor. For example, the production rate of red-antired up quarks will be given by:

$$\sigma(e^+e^- \to u_r\bar{u}_r) = \frac{4\pi}{3}\left(\frac{2}{3}\right)^2 \frac{\alpha^2}{E_{\text{CM}}^2}. \tag{1.2}$$

The charge of an up quark is $2/3$ of that carried by a positron. For notation, let's denote the electromagnetic coupling by Q and the fractional charge by $\mathcal{Q}$, i.e. $Q_u \equiv \mathcal{Q}_u e = \frac{2}{3}e$.

- For the total cross section, we sum over all posssible final states.
  - If the u quark has N_c possible colours, then we sum over all of them such that:

$$\sigma(e^+e^- \to u\bar{u}) = \frac{4\pi}{3}\left(\frac{2}{3}\right)^2 N_c \frac{\alpha^2}{E_{\text{CM}}^2}. \tag{1.3}$$

  - Finally, we assume that the $u\bar{u}$ pair will ultimately produce hadrons, and hence the total rate for producing hadrons is given by summing over all possible quark flavours (with charge factors):

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi}{3}\left(\sum_f \mathcal{Q}_f^2\right) N_c \frac{\alpha^2}{E_{\text{CM}}^2}. \tag{1.4}$$

- Most importantly, if we consider the ratio of hadron production relative to muon production, we have a very simple prediction in terms of quarks:

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_f \mathcal{Q}_f^2. \tag{1.5}$$

Of course this simple prediction should only be valid in the region where $E_{\text{CM}} \gg 2m_f$, for the heaviest flavour of quark (f) that is energetically available.

- To make this concrete, let's first consider $E_{\rm CM} > 2\,{\rm GeV} \gg 2m_s$, but below the threshold for charm production:

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$
$$= N_c\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = \frac{2}{3}N_c. \quad (1.6)$$

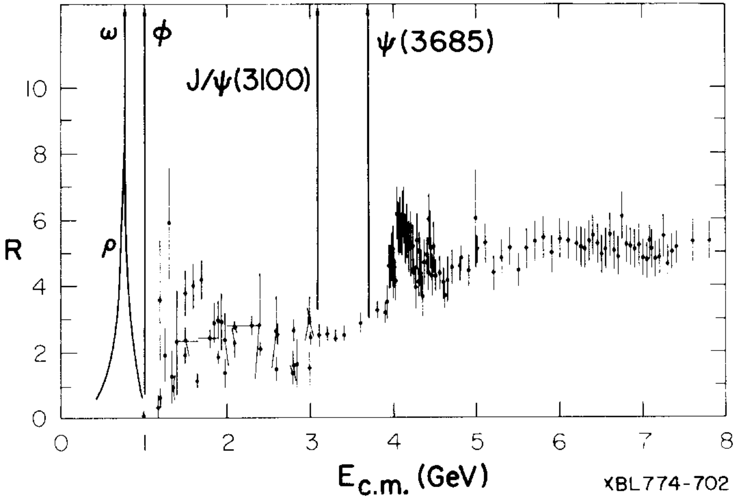
We've not yet talked about the scale of quark masses, but let's take it that the up, down and strange quark masses are much less that 1 GeV.

And, if we go well above charm quark production, we expect this ratio to change to:

Here we're assuming we know that the charge of the charm quark is $\mathcal{Q}_c = +\frac{2}{3}$.

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{10}{9}N_c. \quad (1.7)$$

- Here we take a look at the situation compiled by the Particle Data Group in 1977 (a supplement to the 1976 edition):



Considering our rather simplified assumptions, *the agreement is incredible!*

- In addition to predicting the overall rates of hadron production, the other key feature of this figure shows the distinct line features of the $J/\psi(3100)$ and $\psi(3685)$.
  - If the e^+e^- energies are tuned to just the right value, there is a *massive* increase in the observed cross section.
  - And simply from the Heisenberg uncertainty principle, these very narrow widths correspond to very long-lived states.
- These must correspond to bound states of some *new* type of quark: the **charm** quark.
- As above, we noted that we had a conceptual challenge in the establishment of QCD, where the quark fields of the Lagrangian do not appear as observed particle states.

> If QCD is to be the correct theory of the strong interaction, it must generate a mechanism for quark confinement.

- Fortunately, since the charm quarks are relatively heavy compared to the QCD scale, we can approximate the $c\bar{c}$ interaction in terms of a potential model.

The lifetimes of the J/ψ and ψ are $\mathcal{O}(1{,}000)$ times longer than typical $q\overline{q}$ hadronic states. To quote David Griffiths from his book: *"It's as though someone came upon an isolated village in Peru or Caucasus where people lived to be 70,000 years old. That wouldn't be some actuarial anomaly, it would be a sign of fundamentally new biology at work."*

The treatment of charm quarks in a non-relativistic confining potential was essential for establishing QCD as the fundamental theory, however resolving the nature of confinement for light quark systems remains an active area of research.

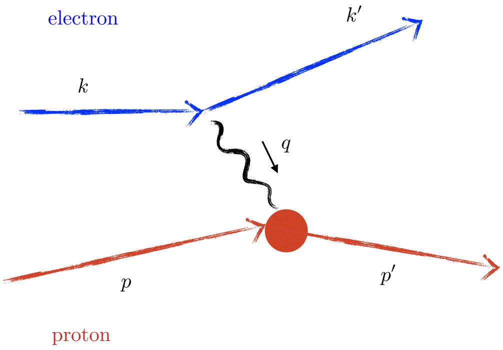
Asymptotic freedom

Although quarks appear to be confined, their interactions appear to be rather weak at high energies.

> The vanishing of the strong coupling constant at (asymptotically) large energies is referred to as asymptotic freedom.

- Tomorrow we will explore the running of the coupling in more detail.
- Here we will highlight the **scaling** phenomena observed in deep-inelastic scattering:
  - In electron–proton scattering at high energies, the proton can be modelled by a distribution of weakly-interacting quarks. *The deviations from non-interacting are given precisely by the predications of QCD.*

Elastic scattering



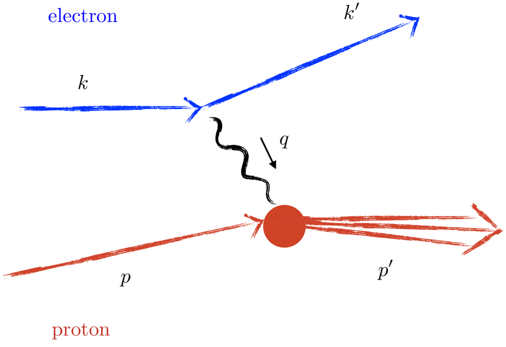
- If we measure the electron scattering angle and final energy, we specify the momentum transfer q completely.
- In the *elastic* case, the final state is a proton, and hence we have

  $$p'^2 = p^2 = m_p^2 = (p+q)^2 = p^2 + q^2 + 2p \cdot q,$$

  and hence we only have a single independent scattering parameter:

  $$Q^2 \equiv -q^2 = 2p \cdot q \tag{1.8}$$

Inelastic scattering



- In the *inelastic* case, we break the proton apart:

$$(p+q)^2 \geq (m_p + m_\pi)^2.$$

- In general, we require two Lorentz scalars to specify scattering, e.g.:

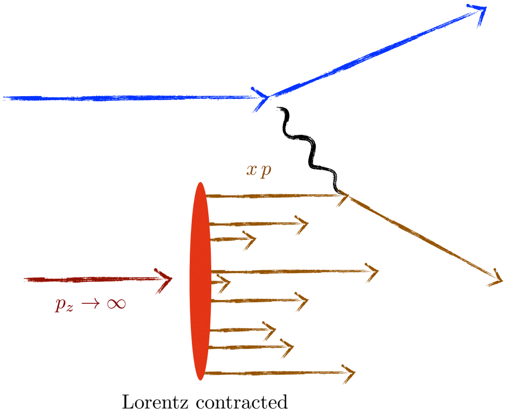
$$W^2 \equiv (p+q)^2, \quad \text{total invariant mass of final-state hadron,} \tag{1.9}$$

$$Q^2 \equiv -q^2, \quad \text{momentum transfer.} \tag{1.10}$$

Parton model

- If we go to large Q^2, we probe the short-distance features of the target.
- Consider a model where a fast-moving proton is a beam of weakly-interacting partons (quarks or gluons)

We consider that we a have a distribution of partons, each carrying a fraction of the momentum of the total momentum of the proton.



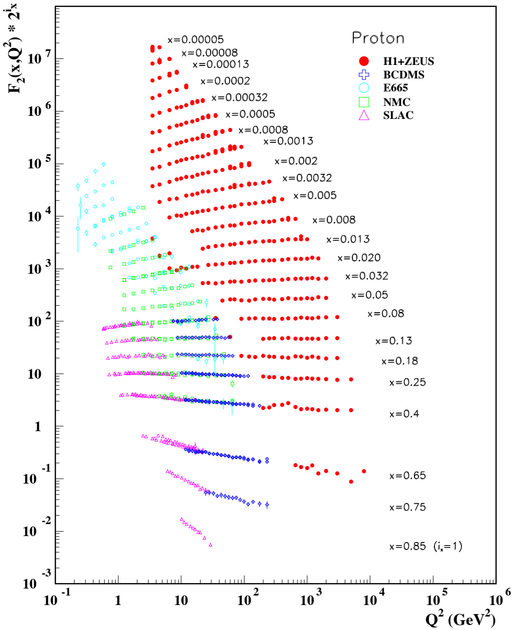
- Suppose we scatter *elastically* from a single quark:

$$(xp)^2 = (xp+q)^2 = (xp)^2 + 2p\cdot q + q^2 \quad \Rightarrow \; 0 = 2x\,p\cdot q + q^2.$$

- In this model the scattering should be specified in terms of the single variable, x: **"Bjorken scaling"**.

In this simplified model, we see that the observed scattering at a particular $x = \frac{2p\cdot q}{-q^2}$, directly identifies the momentum fraction carried by the struck quark. And hence the rate of scattering encodes the probability of finding a quark with that x

Scaling



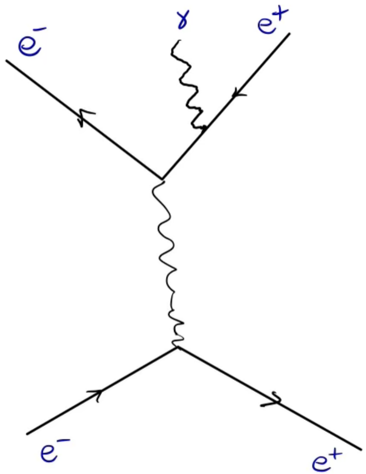
- Bjorken scaling: cross section, at any x, should be independent of Q^2.
- In reality, the observed evolution is weakly dependent on $\log Q^2$. *And QCD predicts this dependence with great accuracy!*

Evidence for gluons: 3-jet events

If we are to describe the interactions between quarks by a gauge theory, we should expect to see evidence for the presence of the gauge bosons: the gluons.

A QED prototype: $ee \to ee\gamma$

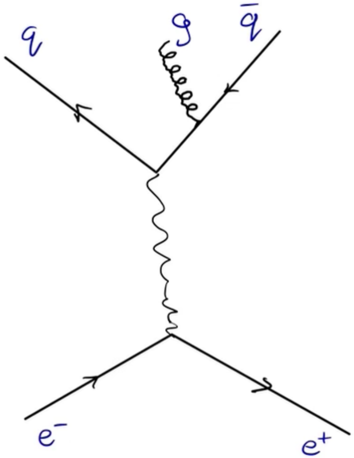
- In QED, we can produce photons in electron-positron collisions: $ee \to ee\gamma$



- We radiate a photon in the collision: "Bremsstrahlung".

Radiating a gluon

- If we have a gauge theory with gluons as the analog of the photon, we should similarly anticipate the process $e^+e^- \to q\bar{q}g$:

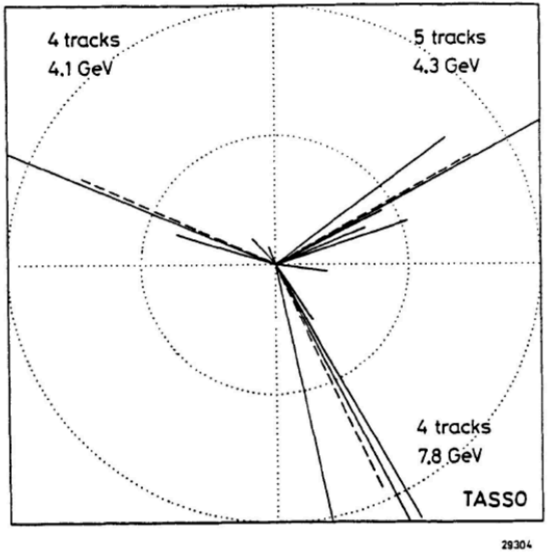


- But we have confinement: quarks and gluons must strip (soft) particles from the vacuum to form colour singlet states.
- However, if each of the $q\bar{q}g$ are hard (i.e. high-energy) we should observe collimated jets of colour-singlet hadrons $\to$ 3-jet events!

By momentum conservation, the 3 jets must lie in a plane, and it will be most easily to identify the signature if each has similar energy and hence they are each separated by 120°.

First observation of 3-jets

- First observation of a 3-jet event was reported at the 1979 Lepton-Photon Conference (Fermi National Accelerator Laboratory).



- Prediction for rate can be made from QCD: *another success!*

1.3 Why were we led to the theory of QCD?

The theory of QCD wasn't created in a vacuum. As experiments probed the high-energy (or short distance) features of hadronic systems, there was a search to develop a consistent theory that could describe a range of puzzling phenomena.

- For context, let's remind ourselves of a brief history of the evolution of particle physics in the lead up to the pre-QCD era:

  1932 The neutron was discovered.

The full progression is of particle physics is of course much richer than summarised here. And we just want to highlight key features that a relevent to understanding the strong nuclear force.

1935–1947 Yukawa proposes pion as carrier of strong force. Eventually disentangled from the similar-in-mass muon.

PROCESSES INVOLVING CHARGED MESONS

By Dr. C. M. G. LATTES, H. MUIRHEAD, Dr. G. P. S. OCCHIALINI and Dr. C. F. POWELL
H. H. Wills Physical Laboratory, University of Bristol

In recent investigations with the photographic method[1,2], it has been shown that slow charged particles of small mass, present as a component of the cosmic radiation at high altitudes, can enter nuclei and produce disintegrations with the emission of heavy particles. It is convenient to apply the term 'meson' to any particle with a mass intermediate between that of a proton and an electron. In continuing our experiments we have found evidence of mesons which, at the end of their range, produce secondary mesons. We have also observed transmutations in which slow mesons are ejected from disintegrating nuclei. Several features of these processes remain to be elucidated, but we present the following account of the experiments because the results appear to bear closely on the important problem of developing a satisfactory meson theory of nuclear forces.

No. 4047 May 24, 1947 NATURE 695

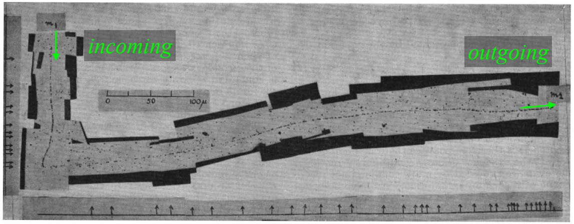


Fig. 1. Observation by Mrs. I. Roberts. Photomicrograph with Cooke × 45 'Fluorite' objective. Ilford 'Nuclear Research', boron-loaded $C2$ emulsion. m_1 is the primary and m_2 the secondary meson. The arrows, in this and the following photographs, indicate points where changes in direction greater than 2° occur, as observed under the microscope. All the photographs are completely unretouched

"...we have found evidence of mesons which, at the end of their range, produce secondary mesons."

And if physics were governed by neat clean principles, we would have been done! Atoms are held together by the Coulomb field, and the compact nuclei of protons and neutrons are held together by the pion field (and neutrinos resolved energy and angular momentum conservation in weak decays). *But...*

1947–1950 The first "strange" meson was found (1947). Cosmic rays strike a lead plate: downstream a pair of charged pions are observed:

$$K^0 \to \pi^+\pi^-, \tag{1.11}$$

and shortly after (1949), another type charged particle is observed to decay to 3 charged pions:

$$K^+ \to \pi^+\,\pi^+\,\pi^-. \tag{1.12}$$

these are clearly heavier than 2 and 3 pions — but still relatively light.

And by 1950, another strange track was seen — something neutral decays to a proton and pion:

$$\Lambda \to p^+\pi^-, \tag{1.13}$$

but this time it must be heavier than $m_p + m_\pi$.

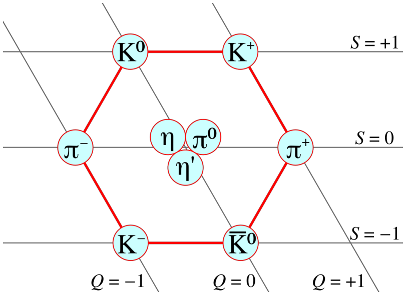
1952 Brookhaven Cosmotron began operation, and so began an era of discovering all types of new states.

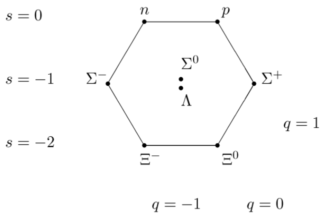
These weren't immediately named kaons, and not was it obvious how similar the K^0 and K^+ are — but it makes our story easier.

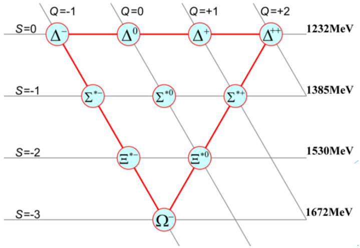
"Mesons" were initially identified by being intermediate in mass, i.e. between an electron and the proton; whereas "baryons" were heavy, at least as heavy as the proton.

A feature of these new states is that they are produced rapidly, $\sim 10^{-23}$ seconds; yet decay slowly, $\sim 10^{-10}$ seconds: there must be different underlying mechanisms at work!

1961–1964 The Eightfold Way

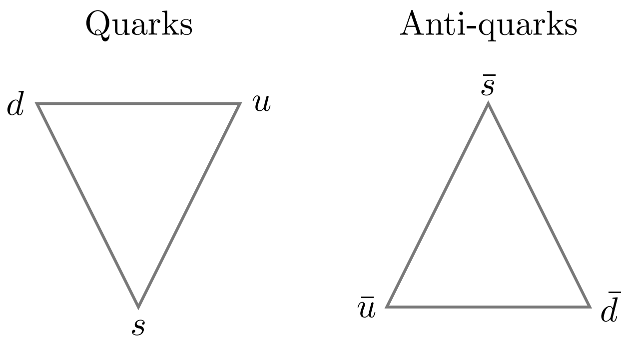






Gell-Mann and Ne'eman (independently) identify a classification scheme for the various new states that were being found: and importantly provide a *prediction* for the Ω^- baryon.

1964 Quarks are proposed as the underlying building blocks that underlie the patterns seen in the eightfold way.



– **Quark model**: All mesons can be identified as $q\bar{q}$ states, and all baryons are qqq.

1964 Within the quark model, single-flavour states such as the Ω^- (sss) or $\Delta^+{+}$ (uuu) appear to be at odds with the Pauli exclusion principle. Greenberg proposes a new quantum number, such that each quark one of a triplet of possible states: **3 colours**.

Gell-Mann introduced the term "colour" in the 1970s.

1962–1964 Gell-Mann (again) identifies current algebra in symmetries of the strong interaction: Symmetries in the vector and axial-vector flavour currents in QCD. Importanly, symmetries are broken, but weakly.

1973 A lot happened:
- The charm discovery,
- Promoting the colour quantum number to a gauge symmetry,
- Non-Abelian gauge theories provide candidate for asymptotically-free theory.

2 QCD formalism

2.1 Non-Abelian gauge theory

Gauge invariance and QED

- From a theoretical point of view, the construction of QCD appears as a very natural extension of the gauge symmetry principle, that has proven so successful for quantum electrodynamics (QED).
- To recap the Abelian case, quantum electrodynamics is based on the invariance of the theory to *local* U(1) transformations:

$$\psi(x) \to \psi'(x) = e^{ie\omega(x)}\psi(x), \tag{2.1}$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu - \partial_\mu \omega(x), \tag{2.2}$$

  where ψ is a fermion field (typically the electron) and A_μ is the corresponding gauge potential, which gives rise to photon excitations.
  Importantly, we note that the phase rotation, $\omega(x)$, is dependent on spacetime.

GWS theory unifies QED with the electroweak force, with the gauge bosons arising from a non-Abelian SU(2) group. Gauge theories don't immediately permit massive gauge bosons, and the Higgs mechanism proved successful in overcoming this.

- It can be shown that the following Lagrangian under this gauge transformation:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - e\gamma^{\mu}A_{\mu} - m\right)\psi, \tag{2.3}$$

with

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}. \tag{2.4}$$

If you've not seen this before, it's worth convincing yourself that this Lagrangian is invariant under the transformation specified above. We find that the partial deriviative in the fermion Lagrangian acts on the phase rotation, and the transformation of A_μ exactly compensates the extra term. And note that it is straghtforward to show that F is invariant under this transformation.

For notation, we'll make use of the covariant derivative, here written as

$$D_\mu = \partial_\mu + ieA_\mu,$$

and Feynman's slash notation, e.g. $\not{D} = \gamma^\mu D_\mu$.

Non-Abelian gauge invariance and QCD

- Extending the gauge principle a non-Abelian group, we first state that the QCD Lagrangian can be expressed compactly by:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_f \bar{\psi}_f \left(i\not{D} - m_f \right) \psi_f, \tag{2.5}$$

  with an implicit sum $a = 1, \ldots, 8$ running over the independent gauge field components, A^a_μ, where we have one for each generator of the symmetry group. We will often write the gauge field in a more compact notation:

$$A_\mu = \sum_{a=1}^{8} A^a_\mu t^a_N, \tag{2.6}$$

  noting that A encodes 4 matrices in colour space — one 3×3 matrix for each spacetime direction.

- The form of the covariant derivative looks similar to that of the Abelian case:

$$D_\mu = \partial_\mu - igA_\mu, \tag{2.7}$$

  but we now identify A_μ as a matrix in colour space, and the ordinary derivative is diagonal in colour, hence this term is implied to be propotional to an identity matrix.

In the current understanding of the Standard Model, there are 6 known flavours of quarks, hence the sum over f runs over u, d, s, c, b, t. In practice, if one is only considering low energies, one only needs to consider the (energetically) active flavours, where the heavy flavours can be integrated out (resulting in a modified coupling for consistency).

- In the above Lagrangian, we've started with the final answer, but where is the gauge invariance that we desired?
- Firstly, the gauge symmetry that we desired is that we can rotate the colour components (for any quark flavour) into each other without changing the physics:

$$\psi_f(x) \to \psi'_f(x) = \Omega(x)\psi_f(x), \tag{2.8}$$

where Ω is a *spacetime-dependent* element of the SU(3) group, which we can write as:

$$\Omega(x) = e^{i\omega^a(x)t^a_N}, \tag{2.9}$$

being parameterised by 8 rotation angles $\omega^a(x)$ — one for each generator.

We can picture ψ_f as a 3-component column vector and Ω as a 3×3 matrix. In terms of colour components, $i, j, = 1, 2, 3$, we can write the transformation in index notation:

$$\psi^i_f \to (\psi'_f)^i = \Omega^{ij}\psi^k_f.$$

The SU(3) generators are commonly represented by the Gell-Mann matrices, up to a factor of 2 in the normalisation, $t^a_N = \lambda^a/2$.

- An easy way to demonstrate the gauge symmetry of the fermionic part of the action is to rewrite the Lagrangian in terms of the transformed fields:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi = \underbrace{\bar{\psi}\Omega^\dagger}_{\bar{\psi}'}\underbrace{\Omega(i\not{D} - m)\Omega^\dagger}_{(i\not{D}'-m)}\underbrace{\Omega\psi}_{\psi'}, \tag{2.10}$$

which clearly represents the symmetry of the theory, *provided* that we identify the tranformation of the covariant derivative:

$$D'_\mu = \Omega(x)D_\mu\Omega^\dagger(x). \tag{2.11}$$

- Given the transformation law for D, we can determine the transformation of the gauge potential:

$$A'_\mu = \Omega A_\mu \Omega^\dagger + \frac{i}{g}\Omega(\partial_\mu\Omega^\dagger). \tag{2.12}$$

For unitary symmetries of interest to us, of course we simply have $\Omega^\dagger\Omega = I$.

The field strength tensor

- We have constructed the formulation of a (locally) gauge invariant action for our "matter" fields. Just as in the Abelian case, we require a description of the kinetic energy in the gauge fields.
- It turns out that we can *define* the non-Abelian field strength tensor in terms of the commutator of covariant derivatives:

$$F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu]. \tag{2.13}$$

It's important to note that the commutator is not a differential operator — it's simply a function that takes a value in the Lie algebra of our symmetry group.

- Given the form of the commutator, it is evident that the field strength tensor is an element of the Lie algebra, and can hence be expressed as:

$$F_{\mu\nu} = F^a_{\mu\nu} T^a. \tag{2.14}$$

Just as for A, F can be expressed independently of the representation.

We can determine the components of F by writing out explicitly in terms of the gauge potential:

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu], \tag{2.15}$$

$$= \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a - igA^a_\mu A^b_\nu [T^a, T^b], \tag{2.16}$$

$$= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu\right) T^a, \tag{2.17}$$

$$\therefore\ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu. \tag{2.18}$$

In the absence of the non-Abelian commutators, you should recognise the usual field strength tensor of QED.

- We could construct the transformation law for F using A, however it is easier to use the commutator form and the covariant derivative transformation ($D' = \Omega D \Omega^\dagger$), giving:

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \Omega(x) F_{\mu\nu}(x) \Omega^\dagger(x). \tag{2.19}$$

- We note that unlike the Abelian $F_{\mu\nu}$, the non-Abelian case is *not* itself invariant under gauge transformations. Of course to construct a gauge invariant action, we need to form products that transform invariantly under the gauge symmetry.

We see that the field strength exhibits a homogeneous transformation law, in constrast to the gauge potential (see Eq. (2.12)).

In the non-Abelian case, F will not generally commute with Ω, hence giving rise to a non-trivial transformation law. Physically, we note that at weak coupling (where we neglect the commutator), F will look like electromagetic $\mathbf{E}$ and $\mathbf{B}$ fields, but having independent components for each of the generators of the group. Since these $\mathbf{E}$ and $\mathbf{B}$ fields are charged under the gauge group, it is not surprising that they will rotate into one another under a gauge tranformation.

The gauge action

- The simplest possible Lagrangian we can construct is built from F^2, where the Lorentz indicies are contracted to form a Lorentz scalar, *and* we take a trace over colour to ensure gauge invariance:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\operatorname{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right). \tag{2.20}$$

The conventional factor of $-\frac{1}{2}$ here is simply chosen to ensure the canonical normalisation in terms of the connection to the Hamiltonian.

- To confirm that this is gauge invariant, we insert the transformation law for F:

$$\mathcal{L}_{\text{gauge}} \to -\frac{1}{2}\operatorname{Tr}\left(F'_{\mu\nu}F'^{\mu\nu}\right), \tag{2.21}$$

$$= -\frac{1}{2}\operatorname{Tr}\left(\Omega F_{\mu\nu}\Omega^\dagger\,\Omega F^{\mu\nu}\Omega^\dagger\right) = \mathcal{L}_{\text{gauge}}, \tag{2.22}$$

where the equality easily drops out by the cyclic property of the trace.

- Finally, to write out in terms of components we use the normalisation of the group generators:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}\,\text{Tr}\left(F^a_{\mu\nu}\,t^a_N\;F^{\mu\nu\,b}\,t^b_N\right), \tag{2.23}$$

$$= -\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu\,b}\underbrace{\text{Tr}\left(t^a_N t^b_N\right)}_{\frac{1}{2}\delta^{ab}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\,a}, \tag{2.24}$$

which agrees with the form we originally introduced in Eq. (2.5).

Interaction terms

- To make the Lagrangian more explicit, as an example we write out the up quark Lagrangian in detail:

$$\bar{\psi}_u^i \left[(i\partial\!\!\!/ - m_u)\delta_{ij} + g A\!\!\!/^{\,a}(t_N^a)_{ij}\right] \psi_u^j, \tag{2.25}$$

  with $i, j = 1, 2, 3$ running over the colour indices and t_N the generators of the (3-dim'l) fundamental representation of SU(3).
- And writing out the gauge part explicitly, we have:

$$-\frac{1}{4} F_{\mu\nu}^a F^{a\,\mu\nu} = -\frac{1}{4} \underbrace{(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2}_{\text{kinetic energy}} - g f^{abc} \underbrace{(\partial^\mu A^{a\,\nu}) A_\mu^b A_\nu^c}_{\text{3-gluon vertex}}$$
$$- \frac{1}{4} g^2 f^{abc} f^{ade} \underbrace{A^{b\,\mu} A^{c\,\nu} A_\mu^d A_\nu^e}_{\text{4-gluon vertex}}. \tag{2.26}$$

2.2 Gauge fixing

Conceptual picture

- In order to construct the Feynman rules of the theory, it is essential to specify a gauge-fixing condition. Just as for the photon in QED, the gluon matrix is highly singular — and hence we are unable to invert to define the propagator. This is a consequence of an unphysical degeneracy associated with the gauge degree of freedom.
- Therefore to define the propagator we prescribe a gauge fixing condition. Choosing a (generalised) Lorenz gauge-fixing condition, $\partial \cdot A = 0$, gives rise to the ξ-dependence of the gluon propagator (see Lect 3 below). This works just the same as for QED.
- In QED, we supplement this gauge-fixed propagator by the condition that only the physical degrees of freedom propagate in external (in/out) states — photons are transverse, i.e. real photons only have two physical polarisations.
  - Internal photon lines sum over all components of the gauge potential, yet it turns out that the unphsical degrees of freedom never contribute to closed loops.

To somewhat oversimplify the issue, A_μ has 4 degress of freedom, but photons only have 2.

- However, the story is a little more complicated in QCD. In this case, additional interaction vertices make it possible for the unphysical degrees of freedom to contribute to loop effects.
  - As it turns out, there is a prescription that we can implement that exactly cancels the unphysical effects in gluon loop graphs; this is known as the **Faddeev-Popov** procedure.
  - The consequence of this is to introduce unphysical "ghost" fields having the properties that they are scalar, Grassmann-valued (anti-commuting) fields charged under the adjoint representation.

It may seem odd to see anti-commuting scalars, since this doesn't match with the statistics of any particles we know of — the ghost fields are purely a mathemetical trick to patch up something unphysical in our treatment of gluons.

Fixing the gauge

- Let's quick recap the issue in QED.
- We consider just the gauge/photon part of the QED action, where we want to do path integrals of the form:

$$\int \mathcal{D}A\, e^{iS[A]}, \quad \mathcal{D}A \equiv \mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2\mathcal{D}A^3. \tag{2.27}$$

Sorry, I'm talking about path integrals before reminding us all what path integrals are. Hopefully we have some familiarity with these by now.

- To highlight the problem, we transform the action to momentum space:

$$S = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + k^\mu k^\nu\right]\tilde{A}_\nu(-k). \tag{2.28}$$

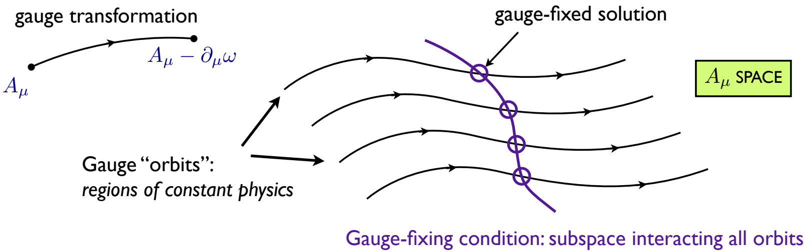
- From the point of view of the path integral, where we wish to integrate over $\mathcal{D}A$, we have a nice simple quadratic form, which should be easy to compute as a Gaussian integral.
- However, we easily identify that this matrix is singular: the action will vanish whenever $\tilde{A}_\mu(k) = k_\mu\tilde{\omega}(k)$, for any $\tilde{\omega}$.
  - With the action vanishing, the integrand of the functional integral is unity over a huge volume of gauge space and hence badly divergent.

- This divergence is a consequence of gauge invariance — we are integrating over a space of physically-equivalent field configurations.
- In the functional integral, we should really consider that we only want to integrate over the physical Hilbert space — any states that are equivalent by a gauge transformation are not independent (and hence shouldn't be double counted).
- By exploiting the gauge symmetry of the theory, we can choose to fix the gauge and hence eliminate the unphysical degrees of freedom.
  - There are many gauge fixing conditions on the market, and each have their advantages depenging on the situation in which they are being used. One common choice is the Lorenz gauge:

$$\partial^\mu A_\mu = 0, \tag{2.29}$$

    which has the attractive feature that it transforms invariantly under Lorentz transformations.

– Whatever our choice of gauge fixing prescription, it can be helpful to consider a pictorial representation:



Each contour depicts a space of gauge-equivalent configurations.

– A suitable gauge fixing condition should have a unique solution for every gauge configuration.

- While straightforward classically, imposing a condition (such as the Lorenz gauge) on quantum field operators is challenging.

Note that Lorenz gauge alone doesn't entirely specify the gauge, and we typically eliminate the final gauge degree of freedom by placing a restriction on the photon polorisation vector.

- The somewhat *ad hoc* prescription that we generally all learn first is to add a term to the photon action that explicitly breaks the gauge degeneracy:

$$S^{GF}_{\text{photon}} \to \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 \right]. \tag{2.30}$$

- Importantly, this breaks the singular nature of the photon matrix, and allows us to invert and hence define the propagator ($\equiv$ Green's function):

$$\Delta_{\mu\nu}(k) = \frac{i}{k^2 + i\varepsilon} \left\{ -g_{\mu\nu} + (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right\}, \tag{2.31}$$

  ... *and then we all run away setting* $\xi = 1$ *and forget we ever went through this story.*

Faddeev-Popov in QED

- Within the path integral formalism, there exists a neat mathematical trick that introduces the gauge-fixing term in the action in a more systematic way — and importantly, gives us a strategy that can more readily be extended to non-Abelian theories.
- The basic idea is that we would like a strategy that would allow us to factorise the functional integral into:

$$\int \text{“gauge transforms”} \times \int \text{“distinct gauge orbits”}.$$

- For the sake of our argument, let's assume that we would like to impose something like the Lorenz gauge-fixing condition: $\partial \cdot \bar{A} = 0$.
- In general, we choose an arbitrary function that we wish to constrain, e.g. $G(\bar{A}) = 0$, and set:

$$G(\bar{A}) = \partial_\mu \bar{A}^\mu - K(x), \tag{2.32}$$

with K being just another arbitrary function that we can't (yet) attach any meaning to.

- To force G to be zero, we can introduce a convenient functional delta function:

$$1 = \int \mathcal{D}G \, \delta[G(\bar{A})]. \tag{2.33}$$

The bar $\bar{A}$ indicates that we have selected a gauge configuration that satisfies the Lorenz condition,

i.e. $\bar{A}^\mu = A^\mu + \partial^\mu \omega$,

where ω is appropriate chosen for any arbitrary A.

- We see that G depends on the gauge transformation:

$$G(\bar{A}) = \partial_\mu A^\mu(x) + \partial^2\omega(x) - K(x), \tag{2.34}$$

and we perform a change of variables, $G \to \omega$:

$$1 = \int \mathcal{D}G\,\delta[G(\bar{A})] = \int \mathcal{D}\omega\ \det\left[\frac{\delta G(\bar{A})}{\delta\omega}\right]\ \delta[\partial\cdot\bar{A} - K]. \tag{2.35}$$

- We've just created a highly-convoluted way of writing the identity, but let's insert this in our original functional integral:

$$\int \mathcal{D}A\mathcal{D}\omega\ \det\left[\frac{\delta G(\bar{A})}{\delta\omega}\right]\ \delta[\partial\cdot\bar{A} - K]\,e^{iS[A]}. \tag{2.36}$$

- The Jacobian is independent of A (in Abelian theory) and can be factored out of the A integral.

- Now we can do a shift of the integration variable $A \to \bar{A}$, with $\mathcal{D}A = \mathcal{D}\bar{A}$ and using the gauge invariance of the action, we have:

$$\int \mathcal{D}A\, e^{iS[A]} = \underbrace{\int \mathcal{D}\omega \; \det\left[\frac{\delta G(\bar{A})}{\delta \omega}\right]}_{\text{factorised!}} \int \mathcal{D}\bar{A}\, \delta[\partial_\mu \bar{A}^\mu - K'] e^{iS[\bar{A}]}, \tag{2.37}$$

  and now $\bar{A}$ is just a dummy variable and we're free to change it back to the label A if we wish. Note that we modified $K \to K'$ in this transformation, but K was arbitrary, so it's just become a new arbitrary function which we'll just go back to using K.
- The $\mathcal{D}\omega$ integral will be some factor, but is independent of A and hence just a constant that will fall out in the normalised ratio, $\mathcal{Z}[J]/\mathcal{Z}[0]$.
- We have now a delta function that restricts $\partial \cdot A$ to the arbitrary function K we introduced earlier.
  - Since K is arbitrary, we can just average over it with a Gaussian:

$$\int \mathcal{D}A\, e^{iS[A]} = N_\xi \int \mathcal{D}K \; \exp\left\{-i \int d^4x \frac{K^2(x)}{2\xi}\right\} \times \int \mathcal{D}\omega \; \det\left[\frac{\delta G(\bar{A})}{\delta \omega}\right] \int \mathcal{D}A\, \delta[\partial_\mu A^\mu - K] e^{iS[A]}. \tag{2.38}$$

As above, the overall normalisation of the functional integrals is irrelevant, however, we've kept a factor of N_ξ just as a normalisation of the Gaussian integral that we've introduced.

– Finally, the δ function allows us to eliminate the K integral and we have:

$$\int \mathcal{D}A\, e^{iS[A]} = N_\xi \int \mathcal{D}\omega \,\det\left[\frac{\delta G(\bar{A})}{\delta\omega}\right] \times \int \mathcal{D}A\, e^{iS[A]} \exp\left\{-i\int d^4x \frac{(\partial\cdot A^2)}{2\xi}\right\}. \quad (2.39)$$

Phew!

- Look at that! We've factored our path integral into some irrelevant constants corresponding to integration over gauge degrees of freedom, and an integral over $\mathcal{D}A$ that has precisely the *ad hoc* gauge-fixing term that we had introduced earlier.

Faddeev-Popov in a non-Abelian theory

- The extension to the non-Abelian case, e.g. QCD, carries through in much the same way as the calculation above. The distinction however is that the gauge transformation is no longer independent of the gauge potential.
- By expanding the gauge transformation defined above (2.12), an infinitessimal gauge transformation can be expressed as:

We've been explicit here to indicate that the covariant derivative is acting in the adjoint representation.

$$\bar{A}^a_\mu = A_\mu + \frac{1}{g} D^{ab}_\mu \omega^b. \tag{2.40}$$

- The argument of the Faddeev-Popov determinant is no longer independent of A. Explicitly, we have:

$$\frac{\delta}{\delta\omega^b(y)}(\partial \cdot \bar{A}^a(x)) = \frac{1}{g}\partial \cdot D^{ab}\delta^{(4)}(x-y). \tag{2.41}$$

- We can represent this functional determinant as a Grassmann Gaussian integral:

If you've seen Grassmann integration before, we remind ourselves that the mutli-dimensional complex Gaussian can be expressed as:

$$\int d\vec{\theta}^* \, d\vec{\theta} \, e^{-\vec{\theta}^* A \vec{\theta}} = \det A.$$

For the ghost fields, we conventionally use $\bar{c}$ for the complex conjugate.

$$\det\left[\frac{\delta G(\bar{A})}{\delta\omega}\right] = \int \mathcal{D}\bar{c}\mathcal{D}c \, \exp\left\{ i \int d^4x \, \bar{c}\,(-\partial \cdot D)\, c \right\}, \tag{2.42}$$

where we have introduced the Grassmann variables $c\bar{c}$, **ghost** fields, to define the functional determinant. Importantly, the operator $\partial \cdot D$ lives has adjoint indices and hence the $c's$ live in the adjoint space — i.e. we have one ghost field for each type of gluon.

- As mentioned earlier, we therefore recognise the ghost fields as Lorentz scalars, but are anticommuting and hence obey Fermi statistics.
- While the determinant factor must be treated explicitly, it doesn't depend on ω and hence the integration over ω still factorises as an overall constant.
- We summarise this by noting that we work with a modified action, that both includes the $1/(2\xi)$ term *and* a new contribution from the ghost fields — and the path integral is extended to include integration over these Grassmann fields.

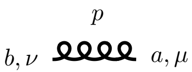
There's a subtlety here, ω only factorised because we considered a perturbatively small gauge transformation, where the gauge transformation, Eq. (2.40), is linear in ω. This will be fine if the gauge fields are small, i.e. in perturbation theory. Going beyond perturbation theory brings us back to the questioning the legitimacy of Eq. (2.35) — but that feels like a story for another day.

3 Running coupling

3.1 Feynman rules

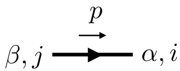
Without derivation, we simply state the Feynman rules of perturbative QCD:

- Gluon propagator



$$= \frac{i\delta_{ab}}{p^2 + i\varepsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]. \tag{3.1}$$

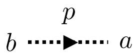
- Quark propagator



$$= \frac{i\delta_{ij}(\not{p} + m_f)_{\alpha\beta}}{p^2 - m_f^2 + i\varepsilon}. \tag{3.2}$$

Just as a reminder on the conventions here: the Greek letters $\mu, \nu, \ldots$ denote Lorentz components; whereas $\alpha, \beta, \ldots$ are understood as Dirac indices. The colour components $i, j, \ldots$ represent components of the fundamental (running over $1, 2, 3$); and $a, b, \ldots$ denote components of the adjoint representation $(1, \ldots, 8)$.

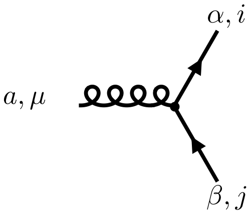
- Ghost propagator



$$= \frac{i\delta_{ab}}{p^2 + i\varepsilon}. \tag{3.3}$$

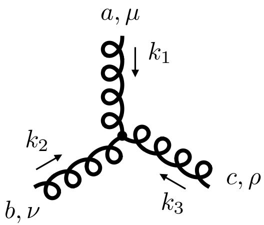
- Quark–gluon vertex

While not derived explicitly, you can see these terms in the interactions described above.



$$= ig\gamma^{\mu}_{\alpha\beta} t^a_{ij} = ig\gamma^{\mu}_{\alpha\beta} \frac{\lambda^a_{ij}}{2}. \tag{3.4}$$

- Triple-gluon vertex

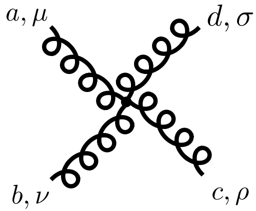


$$= g f^{abc} \left[g^{\mu\nu}(k_1 - k_2)^{\rho} + g^{\nu\rho}(k_2 - k_3)^{\mu} + g^{\rho\mu}(k_3 - k_1)^{\nu} \right],$$

$$\equiv g f^{abc} \Gamma^{\mu\nu\rho}_{g^3}(k_1, k_2, k_3). \tag{3.5}$$

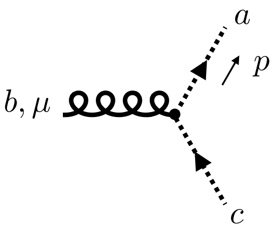
For convenience, we've itroduced the vertex function Γ_{g^3} to collect the Lorentz structure of the 3-gluon vertex.

- Four-gluon vertex



$$= \begin{aligned} -ig^2\big[& f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ & + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ & + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]. \end{aligned} \tag{3.6}$$

- Ghost–gluon vertex

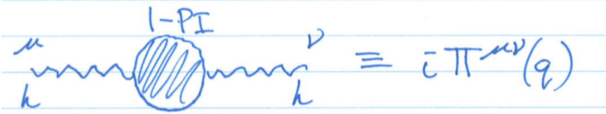


$$= -gf^{abc}p^{\mu}. \tag{3.7}$$

3.2 Vacuum polarisation

Vacuum polarisation in QED (recap)

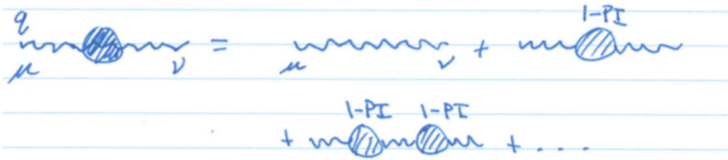
- Let's quickly recap the features of vacuum polarisation in QED (*see also lectures by A Williams*).
- In general, we describe the vacuum polarisation tensor in terms of all 1-particle irreducible corrections to the photon propagator:



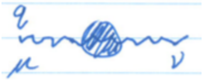
- Since we must satisfy the Ward identity, this tensor can be written in terms of a single scalar function:

$$\Pi^{\mu\nu}(q) \equiv \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi(q^2). \tag{3.8}$$

- The full correction to the photon propagator can be obtained by summing the sequence of 1-PI graphs:



- Because the tensor structure acts as a projector, this sum is rather straightforward to compute as a geometric series:



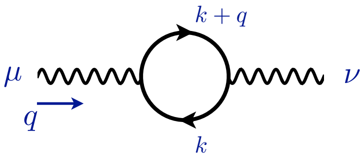
$$= \frac{i}{q^2(1-\Pi(q^2))}\left(-g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - i\frac{q_\mu q_\nu}{q^4}. \tag{3.9}$$

I'm hoping you've seen this or something similar before. In this calculation, we started with Feynman gauge propagtors $\xi = 1$, but have different $q^\mu q^\nu$ terms after resumming the interactions — this is a typical feature and, by gauge invariance, doesn't affect any physical observables.

- Without repeating the details, we'll simply note that the effect of Π is to induce a q^2-dependent scaling of the field normalisation, which ultimately can be absorbed into a running of the effective charge.
- **And how to calculate...**
- We quickly revisit the calculation of Π at one-loop order. Writing down the diagram and evaulating traces, we have:

$$i\Pi^{\mu\nu\,(\text{loop})}_{\text{QED}}(q) = (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \text{tr}\left[\gamma^\mu S_F(k)\gamma^\nu S_F(k+q)\right], \tag{3.10}$$

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}k.(k+q))}{[k^2+i\varepsilon][(k+q)^2+i\varepsilon]} \tag{3.11}$$



Note that the trace identities are unmodified near 4 dimensions. For convenience, we've set the mass of the fermion in the loop to zero, $m \to 0$. One may need to take caution with infrared singularities, but this won't affect the leading UV renormalisation

- We shift to d dimensions and write down the corresponding expressions for the tensor integrals (see the mini-appendix, §3.3):

$$i\Pi^{\mu\nu\,(\text{loop})}_{\text{QED}}(q) = -4e^2\left[2\left(q^\mu q^\nu - \frac{1}{d}g^{\mu\nu}q^2\right)B_2(q^2) + q^\mu q^\nu B_1(q^2) - g^{\mu\nu}q^2 B_1(q^2)\right]. \tag{3.12}$$

and using the ralationships to express in terms of a single scalar function:

$$i\Pi^{\mu\nu\,(\mathrm{loop})}_{\mathrm{QED}}(q) = -2e^2\frac{d-2}{d-1}B_0(q^2)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right). \tag{3.13}$$

- Isolating the scalar function $\Pi(q^2)$, and inserting the expansion for B_0 near $d \sim 4$, we have:

$$\Pi^{(\mathrm{loop})}_{\mathrm{QED}}(q^2) = -\frac{4}{3}\frac{\alpha_{\mathrm{QED}}}{4\pi}\left[\frac{2}{4-d} + \log\frac{-q^2}{\mu^2} + \text{const} + \mathcal{O}(4-d)\right], \tag{3.14}$$

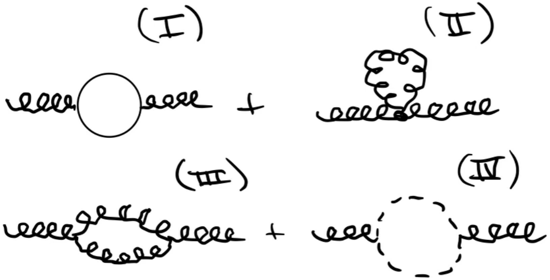
where we've substituted for the fine structure constant, $\alpha_{\mathrm{QED}} = \frac{e^2}{4\pi}$.

For those watching closely, we cancelled the i on the LHS with the i appearing in the expansion of B_0. Also, note that "const" will differ from the constant in the expansion specified in Eq. (3.45).

Gluon self-energy at one loop

- Now let's consider the gluon self energy (QCD vacuum polarisation) at one-loop order, $\mathcal{O}(g^2)$.
- We have the follow diagrams:

$$i\Pi^{\mu\mu'}_{aa'}(q) =$$



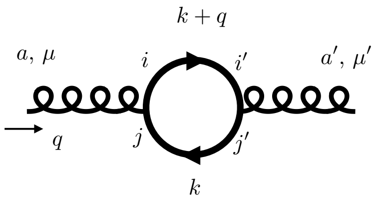
(3.15)

- As in the QED case, gauge invariance dictates that the polarisation tensor is transverse; and colour conservation will also guarantee that the tensor should be diagonal in colour. Hence we should expect to be able to write:

$$i\Pi^{\mu\mu'}_{aa'}(q) = \delta_{aa'}\left(q^2 g^{\mu\mu'} - q^{\mu}q^{\mu'}\right)\Pi(q^2) \tag{3.16}$$

- Let's go through the diagrams one by one...

Quark loop (I)



- This loop looks almost identical to the one in QED above:

$$i\Pi^{\mu\mu'\,(\mathrm{I})}_{aa'}(q) = N_f(-1)(ig)^2 \int \frac{d^4k}{(2\pi)^4}\frac{i\delta_{jj'}}{q^2 - i\varepsilon}\frac{i\delta_{ii'}}{(k+q)^2 + i\varepsilon} \times \mathrm{Tr}\left[\gamma^\mu \not{k} \gamma^{\mu'}(\not{k} + \not{q})\right] t^a_{ij} t^{a'}_{j'i'}, \tag{3.17}$$

where the only appreciable difference is the additional colour factors, which give $\mathrm{Tr}\, t^a t^{a'} = C_F \delta^{aa'} = \frac{1}{2}\delta^{aa'}$, and a sum over flavours (amounting to a factor of N_f which just counts each *massless* quark).

- We hence recycle the solution that we obtained above (3.13):

$$i\Pi^{\mu\mu'\,(\mathrm{I})}_{aa'}(q) = -C_F N_f g^2 \delta^{aa'} \frac{2(d-2)}{d-1} B_0(q^2)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right). \tag{3.18}$$

Yes, $2C_F$ is equal to 1, but we have it as explicit for the moment.

4-point bubble (II)

- Since there is no momentum transfer this loop is simply proportional to

$$\propto \int d^d k \frac{1}{k^2},$$

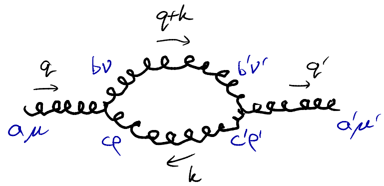
which vanishes in dimensional regularisation — easy, we're done:

$$i\Pi^{\mu\mu'\,(\mathrm{II})}_{aa'}(q) = 0. \tag{3.19}$$

Even if we had momentum tranfer, it would be a single-propagator bubble with no mass term — by translational invariance we could return it's form to the same integral and we'd still conclude that it vanishes.

Gluon loop (III)

- Using our Feynman rules, we write down the gluon loop contribution to the vacuum polarisation:



$$i\Pi^{\mu\mu'\,(\mathrm{III})}_{aa'}(q) = g^2\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i\delta_{bb'}(-g_{\nu\nu'})}{(q+k)^2+i\varepsilon}\frac{i\delta_{cc'}(-g_{\rho\rho'})}{k^2+i\varepsilon}f^{abc}f^{a'b'c'}$$
$$\times\,\Gamma^{\mu\nu\rho}_{g^3}(q,-q-k,k)\Gamma^{\mu'\nu'\rho'}_{g^3}(-q,q+k,-k), \tag{3.20}$$

The $\frac{1}{2}$ factor is a symmetry factor, and we're working in the gauge $\xi = 0$ (Feynman gauge).

where the vertex factors (defined above) are explicitly given by

$$\Gamma^{\mu\nu\rho}_{g^3}(q,-q-k,k) = \left[g^{\mu\nu}(2q+k)^\rho + g^{\nu\rho}(-2k-q)^\mu + g^{\rho\mu}(k-q)^\nu\right], \tag{3.21}$$

$$\Gamma^{\mu'\nu'\rho'}_{g^3}(-q,q+k,-k) = \left[g^{\mu'\nu'}(-2q-k)^{\rho'} + g^{\nu'\rho'}(2k+q)^{\mu'} + g^{\rho'\mu'}(q-k)^{\nu'}\right]. \tag{3.22}$$

- The colour factors also simplify (using our knowledge of Lie algebras), where we have:

$$\delta_{bb'}\delta_{cc'}f^{abc}f^{a'b'c'} = f^{abc}f^{a'bc} = C_A\delta^{aa'}, \tag{3.23}$$

with $C_A = N$ for $SU(N)$.

We make no distinction between *upper* or *lower* colour indices — the location is purely for aesthetics. Sums over repeated indices are assumed.

- Contracting our Lorentz indicies and performing some (tedious, but not complicated) algebra, we find:

In anticipation of performing our loop integrals in dimensional regularisation, we'll do this generally for d dimensions. Here that just means that we're identifying $\mathrm{Tr}\,\delta = d$.

$$\begin{aligned}
&g_{\nu\nu'}g_{\rho\rho'}\Gamma^{\mu\nu\rho}_{g^3}\Gamma^{\mu'\nu'\rho'}_{g^3} \\
&\quad = [g^{\mu\nu}(2q+k)^\rho + g^{\nu\rho}(-2k-q)^\mu + g^{\rho\mu}(k-q)^\nu] \\
&\qquad \times \left[\delta^{\mu'}_{\nu}(-2q-k)_\rho + g_{\nu\rho}(2k+q)^{\mu'} + \delta^{\mu'}_{\rho}(q-k)_\nu\right], && (3.24) \\
&\quad = -g^{\mu\mu'}(5q^2+2k^2+2q\cdot k) + (6-d)q^\mu q^{\mu'} \\
&\qquad + 2(3-2d)k^\mu k^{\mu'} + (3-2d)(q^\mu k^{\mu'} + k^\mu q^{\mu'}), && (3.25) \\
&\quad \equiv N^{\mu\mu'}. && (3.26)
\end{aligned}$$

- Putting these together, we have:

$$i\Pi^{\mu\mu'\,(\mathrm{III})}_{aa'}(q) = -\frac{1}{2}g^2 N_c \delta_{aa'}\mu^{d-4}\int \frac{d^dk}{(2\pi)^d}\frac{N^{\mu\mu'}}{(k^2+i\varepsilon)((q+k)^2+i\varepsilon)}. \tag{3.27}$$

- We can now easily read off the relevant integrals using the results quoted in the mini appendix (Section 3.3),

$$\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{N^{\mu\mu'}}{(k^2+i\varepsilon)((q+k)^2+i\varepsilon)}$$
$$= -g^{\mu\mu'}(5q^2B_0(q^2)+2q^2B_1(q^2))+(6-d)q^\mu q^{\mu'}B_0(q^2)$$
$$+2(3-2d)\left(q^\mu q^{\mu'}-\frac{1}{d}g^{\mu\mu'}q^2\right)B_2(q^2)$$
$$+2(3-2d)q^\mu q^{\mu'}B_1(q^2). \tag{3.28}$$

- Putting all our factors back together, and expressing all in terms of the single scalar function B_0, we have:

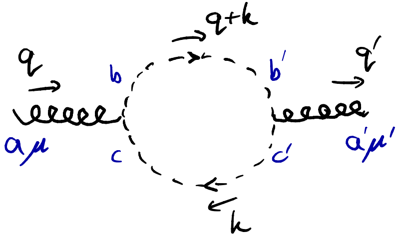
$$i\Pi^{\mu\mu'\,(\mathrm{III})}_{aa'}(q) = g^2N_c\delta^{aa'}\frac{B_0(q^2)}{4(d-1)}\left[(6d-5)q^2g^{\mu\mu'}-(7d-6)q^\mu q^{\mu'}\right]. \tag{3.29}$$

- We immediately recognise that this term on it's own is not transverse — *it doesn't satisfy gauge invariance!* To make this more obvious let's separate off the offending term:

$$i\Pi^{\mu\mu'\,(\mathrm{III})}_{aa'}(q) = g^2N_c\delta^{aa'}\frac{B_0(q^2)}{4(d-1)}\Big[(6d-5)\underbrace{(q^2g^{\mu\mu'}-q^\mu q^{\mu'})}_{\text{transverse}}-(d-1)q^\mu q^{\mu'}\Big], \tag{3.30}$$

where the leftover $q^\mu q^{\mu'}$ term will not satisfy gauge invariance.

Ghost loop (IV)



- The ghost loops act in such a way to cancel the unphysical degrees of freedom that propagated in our gluon loop.
- Using our Feynman rules, we have:

$$i\Pi^{\mu\mu'\,(\text{IV})}_{aa'}(q) = (-1)(-g)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i\delta_{cc'}}{k^2+i\varepsilon} \frac{i\delta_{bb'}}{(k+q)^2+i\varepsilon} \times f^{bac}(k+q)^\mu f^{c'a'b'} k^{\mu'}, \tag{3.31}$$

$$= -g^2 N_c \delta^{aa'} \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^{\mu'} + q^\mu k^{\mu'}}{(k^2+i\varepsilon)((q+k)^2+i\varepsilon)}. \tag{3.32}$$

There's no symmetry factor in this case (the ghost and anti-ghost are distinct). The (-1) factor out the front is from the anticommutation of the ghost fields — just like for a closed fermion loop.

- Using our integrals (§3.3), we obtain:

$$i\Pi^{\mu\mu'\,(\text{IV})}_{aa'}(q) = -g^2 N_c \delta^{aa'} \left[\left(q^\mu q^{\mu'} - \frac{1}{d} g^{\mu\mu'} q^2\right) B_2(q^2) + q^\mu q^{\mu'} B_1(q^2)\right], \tag{3.33}$$

Note we pick up a minus sign from the colour factors here: $f^{bac} f^{ca'b} = -N_c \delta^{aa'}$.

and reducing in terms of the single scalar function B_0 we find:

$$i\Pi^{\mu\mu'\,(\text{IV})}_{aa'}(q) = g^2 N_c \delta^{aa'} \frac{B_0(q^2)}{4(d-1)} \Big[\underbrace{(q^2 g^{\mu\nu'} - q^\mu q^{\mu'})}_{\text{transverse}} + (d-1) q^\mu q^{\mu'}\Big], \tag{3.34}$$

- We can clearly read off that gauge-symmetry violating term exactly cancels that coming from the gluon loop, rendering the sum of the two terms purely transverse.

Renormalisation of gluon propagator

- We complete this part of the calculation by summing up each of the loop contributions:

$$\Pi(q) = \Pi^{(\mathrm{I})}(q) + \Pi^{(\mathrm{II})}(q) + \Pi^{(\mathrm{III})}(q) + \Pi^{(\mathrm{IV})}(q)$$
$$\to i\Pi^{\mu\mu'}_{aa'}(q) = \delta_{aa'}\left(q^2 g^{\mu\mu'} - q^\mu q^{\mu'}\right) g^2 B_0(q^2)$$
$$\times\left(C_A \frac{3d-2}{2(d-1)} - C_F N_f \frac{2(d-2)}{d-1}\right). \tag{3.35}$$

- It is the $\log q^2$ dependence that is most relevant to the running of the coupling, so we'll just summarise this term near $d \sim 4$:

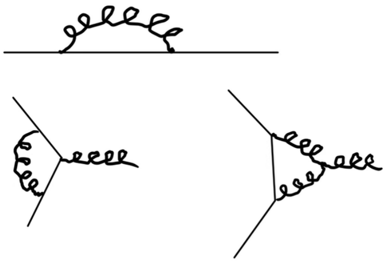
Here we'll explicitly plug back in $C_A = N_c$ and $C_F = 1/2$.

$$\Pi(q^2) \simeq \frac{g^2}{(4\pi)^2}\left(\frac{5}{3}N_c - \frac{2}{3}N_f\right)\log\frac{-q^2}{\mu^2} + \text{poles} + \ldots. \tag{3.36}$$

- Importantly, the gluon loop appears with an opposite sign to the fermion loop contribution. In contrast to the screening phenomena in QED, the gluon loops give rise to **anti-screening.**

Completing the calculation...

- Having (mostly) completed the renormalisation of the gluon propagator, to compute the full running of the coupling we also need to consider the renormalisation of the quark wavefunction and quark–gluon vertex. Hence we need to compute the electron self-energy and the vertex correction... *left as a homework exercise.*
- Graphs:



- For completeness, we'll state the final result for QCD's β function at leading order:

$$\beta(g_R) = -\frac{g_R^3}{16\pi^2}\left[\frac{11}{3}C_A - \frac{4}{3}N_f C_F\right]. \tag{3.37}$$

We've had to skip some of the specifics about implementing a renormalisation scheme, but we'll just note that the β function tells us how the (renormalised) coupling constant changes with renormalisation scale, particularly:

$$\beta = \mu \frac{d}{d\mu} g_R.$$

3.3 Mini-appendix on some massless integrals in dim reg

- In the self-energy graphs above, we encounter loop integrals of the form:

Note the introuction of the scale μ to ensure that the dimensionality of the loop is preserved at arbitrary d.

$$\{B(q), B^{\mu}(q), B^{\mu\nu}(q)\} \equiv \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{\{1, k^{\mu}, k^{\mu} k^{\nu}\}}{(k^2 + i\varepsilon)((k+q)^2 + i\varepsilon)}. \tag{3.38}$$

- We reduce the tensor integrals in terms of scalar functions:

$$B(q) = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + i\varepsilon)((k+q)^2 + i\varepsilon)} \equiv B_0(q^2), \tag{3.39}$$

$$B^\mu(q) = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 + i\varepsilon)((k+q)^2 + i\varepsilon)}, \tag{3.40}$$

$$\equiv q^\mu B_1(q^2) = -\frac{1}{2} q^\mu B_0(q^2). \tag{3.41}$$

$$B^{\mu\nu}(q) = \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 + i\varepsilon)((k+q)^2 + i\varepsilon)}$$

$$\equiv \frac{1}{d} g^{\mu\nu} \overline{B}_2(q^2) + \left(q^\mu q^\nu - \frac{1}{d} g^{\mu\nu} q^2 \right) B_2(q^2), \tag{3.42}$$

$$\overline{B}_2(q^2) = 0, \tag{3.43}$$

$$B_2(q^2) = \frac{d}{4(d-1)} B_0(q^2). \tag{3.44}$$

- And we quote the leading expansion for B_0 near $d = 4$:

$$B_0(q^2) \simeq \frac{i}{(4\pi)^2} \left[\frac{2}{4-d} + \log \frac{-q^2}{\mu^2} + \text{const} + \mathcal{O}(4-d) \right]. \tag{3.45}$$

At spacelike momenta, q^2 will be negative and hence the argument of the log will be near the positive axis, rendering the $i\varepsilon$ irrelevant. We'll ignore it here, but it can be restored with a little patience.

4 Lattice QCD

Notes inspired by P Shanahan (MIT): *"2022 TMD Winter School"*.

Any errors would be mine.

4.1 Euclidean, discretised path integrals

Euclidean path integrals

- Generating functional contain all information about the theory:

$$\mathcal{Z}[\underbrace{J,\eta,\bar\eta}_{\textit{sources}}] = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi \, \exp\left[i\int d^4x\,(\mathcal{L}_{\mathrm{QCD}} + J_\mu A^\mu + \bar\eta\psi + \overline{\psi}\eta)\right]. \tag{4.1}$$

The integrations, $\mathcal{D}A\ldots$, represent integrals over all possible gauge and fermion fields — i.e. something like all possible *paths* through configuration space.

The fermion fields must be anticommuting to ensure that they obey Fermi-Dirac statistics and hence are represented as Grassman numbers.

- Correlation functions expressed as derivatives of $\mathcal{Z}$ with respect to sources:

$$\langle 0|A^\mu(x)\overline{\psi}(y)\cdots\overline{\psi}(z)|0\rangle = \frac{1}{\mathcal{Z}[0]}\frac{\delta}{\delta J_\mu(x)}\cdots\frac{\delta}{\delta\bar\eta(z)}\mathcal{Z}[J,\eta,\bar\eta]\bigg|_{J=\eta=\bar\eta=0}. \tag{4.2}$$

- In order to compute, we must define the path integration measure appropriately, and regularise the action.
  - *Perturbation theory*: Compute the quadratic (Gaussian) part directly and expand correlators in powers of the interaction couplings.
  - *Lattice*: Define $\mathcal{Z}$ directly through lattice regulator, valid beyond perturbation theory, provides a numerical evaluation approach.

- Integrand from $e^{-iS_{\rm QCD}}$ is highly oscillatory, such that the evaluation requires delicate cancellations between different regions of phase space $\rightarrow$ difficulties for numerical implementation.
  $\Rightarrow$ Solution is to Wick rotate ($t \rightarrow -it_{\mathcal{E}}$) from Minkowski to Euclidean space-time. Gives a probabalistic interpretation to functional integral:

$$e^{-iS^{\mathcal{M}}_{\rm QCD}} \rightarrow e^{-S^{\mathcal{E}}_{\rm QCD}}.$$

The exponential is exactly the Boltzmann weighting of a statistical ensemble.

Basic elements of lattice QCD

- Discretise theory onto e.g. 4D cubic lattice of spacing a.
  - The action integral is rewritten as a discrete sum:

$$\int d^4x \to a^4 \sum_{\Lambda}, \tag{4.3}$$

    with spacetime being represented by the discrete set of points:

In practice, lattice are of finite extent by imposing boundary consitions, e.g. (anti-)periodic.

$$\Lambda = \left\{ x \in \mathbb{R}^4 | x = an, n \in \mathbb{Z}^4 \right\}. \tag{4.4}$$

  - The quark fields take values on the lattice sites, $\psi(x) \to \psi(an)$, with $n \in \mathbb{Z}^4$.
  - And (as we will soon see), in order to maintain gauge invariance in the discretised theory, it is convenient to define gauge link variables,

$$A_\mu(x) \to U_\mu(an) = \exp\left(-iagA_\mu(an)\right). \tag{4.5}$$

  - The integrations over paths are replaced by the integration over all values of the fields at each site (or link):

$$\text{e.g.} \int \mathcal{D}\psi \to \int \prod_{x \in \Lambda} d\psi(an).$$

- Any operator of interest is then expressed as:

$$\langle\Theta\rangle = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{-S_{\mathrm{QCD}}}[U,\psi,\overline{\psi}]\Theta[U,\psi,\overline{\psi}]. \tag{4.6}$$

The operator here can be any possible product of fields at arbitrary spacetime points— though in practice, we are mostly interested in 2-, 3-point functions of colour singlet (i.e. gauge invariant) operators.

- Since the action takes the form $S_{\mathrm{QCD}} = S_{\mathrm{glue}} + \int \overline{\psi}\mathcal{M}\psi$, the integration over the fermion fields is Gaussian and can be done exactly. The dependence on $\psi, \overline{\psi}$ in Θ are then replaced by corresponding Wick contractions of the field operators, resulting in factors of the inverse of the Dirac matrix:

$$\langle\Theta\rangle = \frac{1}{\mathcal{Z}_0}\int \mathcal{D}U\ \det\mathcal{M}[U]e^{-S_{\mathrm{glue}}[U]}\Theta[U,\mathcal{M}^{-1}[U]]. \tag{4.7}$$

- The field integrations are reduced to just integrating over the gauge links.
- The operator Θ will be specific to the quantity of interest, will the integrand having a common weight factor:

  $e^{-S_{\mathrm{glue}}[U]}\det\mathcal{M}[U]$

  - We sample to the space according with this weight as a probability measure.
  - Do this via a Markov chain process $U^{[0]} \to U^{[1]} \to \ldots$ with transitions required to satisfy conditions such that the desired probability distribution emerges as equilibrium distribution of the process.

- Once a representative set of configurations is available, observables can be computed as simple averages:

$$\langle \Theta \rangle = \sum_{i}^{N_{cfg}} \Theta[U^{[i]}] + \mathcal{O}\left(\frac{1}{\sqrt{N_{cfg}}}\right), \tag{4.8}$$

  where the error term represents the fact that we only have a finite statistical representation of the full path integral.

- Complete calculations require dealing with:
  - Statistical uncertainties: $N_{cfg} \to$ "large".
  - Systematic uncerainties: multiple lattice spacings (couplings), box sizes, etc.

4.2 Defining lattice actions

- In continuum field theory, in order to construct a gauge symmetry, we were forced to demand the existence of a gauge field in order to define the covariant derivative $\to$ the same is true in a lattice formulation.
- Consider the Euclidean Dirac action for a free fermion:

$$\int d^4x\, \overline{\psi}(x)\, [\partial\!\!\!/ + m]\, \psi(x). \tag{4.9}$$

If we discretise the derivative (e.g. by the symmeteric finite difference) we obtain the replacement:

From now on, I'll mostly be setting the factors of the lattice spacing to 1, $a \to 1$. We'll restore a from time to time when we wish to be explicit.

$$\overline{\psi}(x)\partial\!\!\!/\psi(x) \to \frac{1}{2}\sum_{\mu} \left[\overline{\psi}(n)\gamma_\mu\psi(n+\hat{\mu}) - \overline{\psi}(n)\gamma_\mu\psi(n-\hat{\mu})\right], \tag{4.10}$$

and we observe that the products involve field operators that are spatially separated.

- These terms cannot possibly transform invariantly under *local* gauge transformations, e.g.

$$\psi(n) \to \Omega(n)\psi(n), \quad \overline{\psi}(n) \to \overline{\psi}(n)\Omega^\dagger(n), \quad \Omega(n) \in SU(3).$$

- In order to obtain a loal gauge symmetry, we must introduce "gauge links" as parallel transport operators, U_μ, which have the transformation law:

$$U_\mu(n) \to \Omega(n) U_\mu(n) \Omega^\dagger(n + \hat{\mu}), \tag{4.11}$$

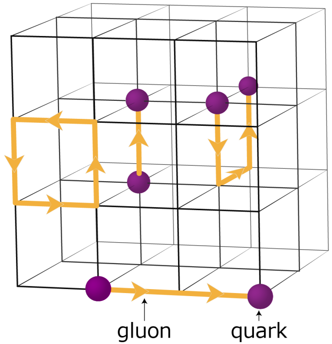
  and we replace the terms in the naive finite difference with terms involving the gauge links:

$$\overline{\psi}(n)\gamma_\mu \psi(n + \hat{\mu}) \to \overline{\psi}(n)\gamma_\mu U_\mu(n)\psi(n + \hat{\mu}), \tag{4.12}$$

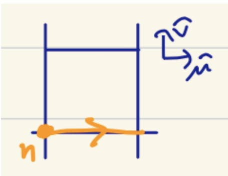
  which automatically satisfies the gauge desired gauge invariance.
- These gauge links are a special case of parallel transport operators that map coordinates in an internal symmetry space from one point to another:

$$U_\mu(n) = \mathcal{P} \exp \left[i \int_0^a d\lambda A_\mu(n + \lambda\hat{\mu}) \right] = \exp iaA_\mu, \tag{4.13}$$

$$\to 1 + iaA_\mu(n) + \mathcal{O}(a^2), \text{ for small } a. \tag{4.14}$$



- We hence identify the gauge links on the *edges* of the lattice:



$$U_\mu(n) = U_{-\mu}^\dagger(n + \hat{\mu}).$$

- Hence we construct the "naive" fermion action:

$$
\begin{aligned}
S_F[U,\psi,\overline{\psi}] &= \frac{1}{2}\sum_{n\in\Lambda}\overline{\psi}(n)\left\{\gamma_\mu\left[U_\mu(n)\psi(n+\hat{\mu}) - U^\dagger_\mu(n-\hat{\mu})\psi(n-\hat{\mu})\right]\right\}, \\
&\equiv \sum_{n,m\in\Lambda}\overline{\psi}(n)\mathcal{M}^{\text{naive}}_{nm}[U]\psi(m), \qquad (4.15)
\end{aligned}
$$

with $\mathcal{M}^{\text{naive}}_{nm}[U]$ defining the naive fermion matrix.

Choices of fermion action

- The Taylor expansion of U_μ, ψ in a gives $\mathcal{O}(a^2)$ errors. BUT "doubling problem": First-order derivative only couples sites separated by $2a \to$ in the continuum limit there are $2^d = 16$ quark flavours instead of 1.
- *Nilsen-Ninomiya no-go theorem*: It is not possible to construct a lattice fermion action that is (ultra)-local, chirally symmetric, free of doublers & have the correct continuum limit.
- We make a choice about how implement the discretisation of the fermion action:

Wilson fermions: breas chiral symmetry explicity (even when $m = 0$) by adding a second derivative term that acts as a large mass (on the order of the cutoff) at the doubler poles, but is irrelevant at zero momentum — i.e. it drives the doublers to higher energies by introducing a term in action that vanishes as $a \to 0$.

Staggered fermions: distribute the 4 components of teh Dirac spinor to different lattice sites $\to$ 4 species or "tastes", break taste symmetry.

Ginsparg-Wilson fermions: Preserve "lattice chiral symmetry", i.e. Ginsparg-Wilson relation, the reduces to usual chiral symmetry in the chiral limit. Various examples:

– Domain-wall fermions (Kaplan & Shamir)

– Overlap fermions (Narayanan & Neuberger)

– Perfect actions / fixed-point fermions (Hasenfratz et al.)

BUT computationally more expensive.

- *Which fermion action is best can depend on the particular application.*

Gauge action

- Closed paths of gauge links (Wilson loops) can be used to construct all gauge-invariant quantities involving only gague fields.
- As for fermions, precise choice of gauge action is irrelevant if it has the correct continuum limit.
- The simplest closed loop is the 1×1 plaquette:

$$P_{\mu\nu} = \mathrm{Re}\,\mathrm{Tr}\left[U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n)\right], \tag{4.16}$$

which we can Taylor expand the path-ordered expression for U_μ in terms of A_μ to identify:

$$P_{\mu\nu} \simeq 1 - \frac{1}{2}g^2\,\mathrm{Tr}\left\{[F_{\mu\nu}(n)]^2\right\} + \mathcal{O}(g^2a^2, a^4, g^4a^2). \tag{4.17}$$

- Allowing us to write the **Wilson gauge action**:

$$S^W_G[U] = \frac{2}{g^2}\sum_n\sum_{\mu<\nu}[1 - P_{\mu\nu}(n)] \to \frac{1}{2g^2}\sum_n\sum_{\mu<\nu}\mathrm{Tr}\left[F^2(n)\right]. \tag{4.18}$$

- Can "improve" the gauge action by including additional loops (e.g. 1×2 rectangels) with coefficients tuned to remove leading discretisation artefacts. Names like "Iwasaki", "tree-level improved" gauge actions etc.

- Now we write the total lattice action as

$$S_{\text{QCD}} = S_G[U] + S_F[U, \psi, \overline{\psi}]. \tag{4.19}$$

Quick check on symmetries:

- SU(3) gauge: preserved by lattice actions.
- Lorentz: broken down to hypercubic $H(4) \Rightarrow$ induces operator mixing.
- Chiral: depends on (fermion) action.

4.3 Correlators and the spectrum

What can be computed in LQCD are Euclidean n-point correlation functions $\rightarrow$ relate physics of interest to matrix elements of local (or non-local) operators.

Example 1: Mass of the pion

- We can calculate the mass of the pion from 2-point correlation function.
- Consider two equivalent expresseions for hadron 2-point currelator:

$$C_\pi(\mathbf{x}, t) \equiv \langle 0|\theta_\pi(\mathbf{x}, t)\theta_\pi^\dagger(\mathbf{0}, 0)|0\rangle, \tag{4.20}$$

Trace in Hilbert space (transfer matrix):

$$= \frac{1}{\mathcal{Z}} \operatorname{Tr}\left[e^{-\hat{H}(T-t)}\hat{\theta}_\pi(\mathbf{x})e^{-\hat{H}t}\hat{\theta}_\pi^\dagger(\mathbf{0})\right]], \tag{4.21}$$

Path integral:

$$= \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-S_{\text{glue}}-\int_V \overline{\psi}\mathcal{M}\psi}\theta_\pi(\mathbf{x}, t)\theta_\pi^\dagger(\mathbf{0}, 0) \tag{4.22}$$

- The operator $\theta_\pi[U,\psi,\overline{\psi}]$ is an *"interpolating operator"* with the quantum numbers of the state of interest, e.g. for a pion:

$$\theta_\pi(x) = \overline{u}(x)\gamma_5 d(x). \tag{4.23}$$

- Firstly, starting from Eq. (4.21, we rearrange to solve for the t-dependence of $C_\pi(\mathbf{x},t)$:

$$\begin{aligned} C_\pi(\mathbf{x},t) &= \frac{1}{\mathcal{Z}}\,\mathrm{Tr}\left[e^{-\hat{H}(T-t)}\hat{\theta}_\pi(\mathbf{x})e^{-\hat{H}t}\hat{\theta}^\dagger_\pi(\mathbf{0})\right], &(4.24)\\ &= \frac{1}{\mathcal{Z}}\sum_\rho \langle\rho|e^{-\hat{H}(T-t)}\hat{\theta}_\pi(\mathbf{x})e^{-\hat{H}t}\hat{\theta}^\dagger_\pi(\mathbf{0})|\rho\rangle, &(4.25)\\ &= \frac{1}{\mathcal{Z}}\sum_{\rho,\sigma} \langle\rho|e^{-\hat{H}(T-t)}\hat{\theta}_\pi(\mathbf{x})|\sigma\rangle\langle\sigma|e^{-\hat{H}t}\hat{\theta}^\dagger_\pi(\mathbf{0})|\rho\rangle, &(4.26)\\ &= \frac{1}{\mathcal{Z}}\sum_{\rho,\sigma} e^{-E_\rho(T-t)}e^{-E_\sigma t}\langle\rho|\hat{\theta}_\pi(\mathbf{x})|\sigma\rangle\langle\sigma|\hat{\theta}^\dagger_\pi(\mathbf{0})|\rho\rangle. &(4.27)\end{aligned}$$

I'll use T to denote the full temporal extent of the lattice — where (anti-)periodic boundary conditions are used for the (quark)gluon fields.

- Similarly,

$$\begin{aligned}\mathcal{Z} &= \sum_\rho \langle\rho|e^{-\hat{H}T}|\rho\rangle = \sum_\rho e^{-E_\rho T}, &(4.28)\\ &\simeq e^{-E_0 T}\left(1 + e^{-\Delta E_1 T} + e^{-\Delta E_2 T} + \dots\right), &(4.29)\end{aligned}$$

where we've used $\Delta E_n = E_n - E_0$ and vacuum energy E_0.

- By taking the limit T becomes large, only $|\rho\rangle = |0\rangle$ contributes to Eq. (4.27) and $\mathcal{Z}$, and hence

$$C_\pi(\mathbf{x},t) \overset{T\to\infty}{\to} \sum_\sigma \langle 0|\hat{\theta}_\pi(\mathbf{x})|\sigma\rangle\langle\sigma|\hat{\theta}_\pi^\dagger|0\rangle\, e^{-\Delta E_\sigma\, t}, \tag{4.30}$$

  and we can project onto zero momentum by taking a sum over $\mathbf{x}$:

$$C_\pi(t) = \sum_{\mathbf{x}} C_\pi(\mathbf{x},t) = \sum_{\sigma(\mathbf{p}=0)} |Z_\sigma|^2\, e^{-m_\sigma t}. \tag{4.31}$$

- We see that this correlator depends of the energies of all states for which $Z_\sigma \neq 0$ — i.e. those that can be created from the vacuum by the creation operator, $\theta_\pi^\dagger(\mathbf{x},t)$.
- In general, a creation operator creates a state that is a linear superposition of all possible eigenstates of H that have the same quantum numbers as the pion. For example: pion, excitations of the pion, three pions in $J=0, I=1$ state, etc.
  However, particular operators can have "stronger" overlaps onto some states compared to others.

- Now let's see how to use the path integral representation to compute C_π in LQCD:

$$C_\pi(\mathbf{x},t) = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{-S_{\text{glue}}-\int_V \overline{\psi}\mathcal{M}\psi}$$
$$\times\, \overline{\psi}_u(\mathbf{x},t)\gamma_5\psi_d(\mathbf{x},t)\,\overline{\psi}_d(0)\gamma_5\psi_u(0), \tag{4.32}$$
$$= \int \mathcal{D}U \det \mathcal{M}_u \, \det \mathcal{M}_d \, e^{-S_{\text{glue}}}$$
$$\times\, \text{Tr}\left[\mathcal{M}_u^{-1}(x,0)\gamma_5\mathcal{M}_d^{-1}(0,x)\gamma_5\right], \tag{4.33}$$
$$\to \frac{1}{N_{\text{cfg}}}\sum_{j=1}^{N_{\text{cfg}}} \text{Tr}\left[\mathcal{M}_u^{-1}[U_j]\gamma_5\mathcal{M}_d^{-1}[U_j]\gamma_5\right], \tag{4.34}$$

where the ensemble of gauge configurations is $\{U_1,\ldots,U_{N_{\text{cfg}}}\}$. And, as above, we sum over all (spatial) positions $\mathbf{x}$ to project onto the zero-momentum correlator, $C_\pi(t)$.

The quark propagators, $\mathcal{M}^{-1}$, are computed by matrix inversion.

- Having computed $C_\pi(t)$ numerically, we can fit the functional form from the transfer matrix to determine the mass of the lightest state with the given quantum numbers.
  - Specifically, we write:

$$C_\pi(t) = |Z_0|^2 e^{-m_0 t} + |Z_1|^2 e^{-m_1 t} + \dots . \tag{4.35}$$

  - Often, it can be helpful to visualise the numerical results in terms of an "effective mass":

$$m_{eff}(t) = \frac{1}{\delta} \log \frac{C_\pi(t)}{C_\pi(t+\delta)} = m_0 + A e^{-\Delta m\, t} + \dots , \tag{4.36}$$

    with Δm being the gap to the first excited energy level in the channel of interest.

– We estimate m_0 as the average of points in the “plateau” region.
– More sophisticated: fit to (multi)-exponential functional form (χ^2 minimisation etc.). Note that it’s essential to take consider correlations in underlying statistical ensemble.

Example 2: Matrix elements, g_A

- We'll consider the isovector axial coupling of the nucleon — this is the QCD part that dominates calculation of the neutron beta decay.
- The matrix element of interest is defined by

$$\langle N(\mathbf{p})|\underbrace{\overline{\psi}\gamma_\mu\gamma_5\tau_3}_{A_\mu^3}|N(\mathbf{p})\rangle = g_A \overline{u}_p(\mathbf{p})\gamma_\mu\gamma_5 u_p(\mathbf{p}). \tag{4.37}$$

- We relate the desired matrix element to a (Euclidean) 3-point function:

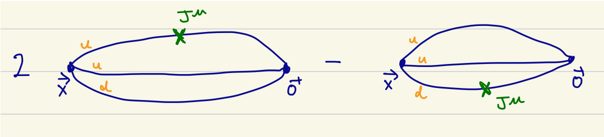
$$C_\mu^3(t,\tau) = \frac{1}{\mathcal{Z}}\sum_{\mathbf{x},\mathbf{y}} \mathrm{Tr}\left[e^{-\hat{H}(T-t)}\theta_N(\mathbf{x},t)e^{-\hat{H}(t-\tau)}A_\mu^3(\mathbf{y},\tau)e^{-\hat{H}\tau}\theta_N^\dagger(0)\right], \tag{4.38}$$

$$\xrightarrow{T\to\infty} \sum_{\mathbf{x},\mathbf{y}}\sum_{n,m}\langle 0|\hat{\theta}_N|n\rangle\langle n|e^{-\hat{H}(t-\tau)}A_\mu^3(\mathbf{y},\tau)e^{-\hat{H}\tau}|m\rangle\langle m|\hat{\theta}_N^\dagger|0\rangle, \tag{4.39}$$

$$= \sum_{\mathbf{x},\mathbf{y}}\sum_{n,m} Z_n Z_m^\dagger\, e^{-E_n(t-\tau)}e^{-E_m\tau}\langle n|A_\mu^3|m\rangle. \tag{4.40}$$

- At large times (in both times), we saturate to the ground states, and hence isolate the matrix element on interest.

- We use the 2-point functions with same θ_N to determine Z_n and E_n. In practice, it is often useful to construct ratios of 3-pt/2-pt to eliminate leading time dependence before fitting.
- For the lattice calculation, we construct the same 3-point function and express in terms of appropriate quark propagators (from Wick contraction field operators). Often we'll represent this in terms of "skeleton diagrams":



A taste of the things that can be calculated

2-pt functions spectroscopy (and including resonant excitations), decay constants,...

3-pt functions [local operators] g_A, scalar charges (for dark matter σ), nucleon electromagnatic form factors, moments of parton distributions, B–π decay form factors to constrain CKM matrix elements...

3-pt functions [non-local operators] quasi/pseudo PDFs, transverse momentum distributions, ...

4-pt functions double-β decay matrix elements, K–$\bar{K}$ mass difference, hadron tensor, (virtual) Compton amplitude, ...

Addtional reading on LQCD:

- There are some many good lecture notes available, including:
  - Lepage: “Lattice QCD for novices”, hep-lat/0506036.
  - Gupta: “Introduction to Lattice QCD”, hep-lat/9807028
  - L”uscher, “Computational strategies in LQCD”, arXiv:1002.4232 [hep-lat].
- Some more recent(ish) texts:
  - Knechtli, Günther & Peardon: “Lattice Quantum Chromodynamics: Practical Essentials”
  - Gattringer & Lang: “Quantum Chromodynamics on the Lattice”
  - DeGrand & DeTar: “Lattice Methods for Quantum Chromodynamics”
- And the classics texts:
  - Creutz: “Quarks, gluons and lattices” (reissued Open Access in 2022)
  - Montvay & Münster: “Quantum fields on the lattice”
  - Rothe: “Lattice gauge theory”