Group Theory:

a brief introduction to some elements thereof

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Lecture 3: Representations of Lie groups and algebras

Definition of matrix representation

The algebraic pattern of $\mathfrak{su}(2)$ appears in a number of different contexts. Although we started with 2×2 hermitean traceless matrices, this is not essential to its structure.

Representation theory studies how a group can be expressed using matrices from $GL(n, \mathbb{C})$. ⟨why do we work over $\mathbb{C}$?⟩

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A *representation of dimension n* defines a matrix $D(g) \in GL(n, \mathbb{C})$ for every $g \in G$ so that the mapping $D : G \to GL(n, \mathbb{C})$ is a homomorphism. In particular:

- Identity: $D(I) = I_n$.
- Inverses: $D(g^{-1}) = [D(g)]^{-1}$.
- Products: $D(g * h) = D(g)D(h)$.

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- Products: $D(g * h) = D(g)D(h)$.

Two representations, D_a, D_b are *equivalent* if there is a single matrix $S \in GL(n, \mathbb{C})$ so that $S^{-1}D_a(g)S = D_b(g)$ for all $g \in G$.

Terminology of matrix representations

- A representation is *faithful* if $D(g) = I_n$ if and only if $g = I \in G$.
- The *trivial* representation is the map that sends every element to the identity: $D(g) = I$.
- A subspace $W \subset \mathbb{C}^n$ is *invariant* if $D(g)\vec{w} \in W$ for all $g \in G$ and $\vec{w} \in W$. ⟨how do you find invariant subspaces?⟩
- An *irreducible* representation is one with no non-trivial invariant subspaces.
- A *reducible* representation is equivalent to one in a fixed block-triangular form: $D_{n+m}(g) = \begin{pmatrix} D_n(g) & R_{nm}(g) \\ 0 & D_m(g) \end{pmatrix}$.
  The first n-coordinates of $\mathbb{C}^{n+m}$ are an invariant subspace for D_{n+m}.

- A *completely reducible* representation is equivalent to a block-diagonal form:

$$D_n(g) = \begin{pmatrix} D_{n_1}(g) & 0 & \cdots & 0 \\ 0 & D_{n_2}(g) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & D_{n_r}(g) \end{pmatrix}$$

  where each D_{n_k} is an *irreducible* representation for the group G. The subscripts here refer to dimension so $\sum_k n_k = n$. This means the coordinates spanning each block form distinct invariant subspaces.

- A completely reducible representation is equivalent to the *direct sum* of the irreducible representations in its diagonal blocks, written as $D_n = D_{n_1} \oplus D_{n_2} \oplus \cdots D_{n_r}$.

Suppose $D : G \to GL(n, \mathbb{C})$ is a representation of a matrix Lie group G. Then there is a unique representation $D' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ such that $D(e^{iT}) = e^{iD'(T)}$. We compute $D'(T)$ as $D'(T) = \frac{d}{dt} D(e^{itT})\big|_{t=0}$.

This definition ensures the matrices $D(e^{iT})$ and $D'(T)$ are expressed with respect to the same basis for $\mathbb{C}^n$.

Note that $\mathfrak{gl}(n, \mathbb{C})$ is a vector space of matrices with matrix commutation as Lie bracket. In general, a representation of a Lie algebra is a homomorphism that maps the Lie bracket of $\mathfrak{g}$ to matrix commutation in $\mathfrak{gl}(n, \mathbb{C})$.

Suppose $D' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ is a Lie algebra representation. Then setting $D(e^{iT}) = e^{iD'(T)}$ will give a representation of the connected and simply-connected covering group G associated with the Lie algebra $\mathfrak{g}$.

Given $D' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ define a representation of $\mathfrak{g}_{\mathbb{C}}$ to be $D'_{\mathbb{C}}(X + iY) = D'(X) + iD'(Y)$. Conversely, every representation of $\mathfrak{g}_{\mathbb{C}}$ becomes a representation of $\mathfrak{g}$ because $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$.

The following theorems tell us that for certain cases a finite-dimensional representation can be built as the direct sum of irreducible representations

If G is a compact matrix Lie group then every finite dimensional representation is completely reducible.

If G is a matrix Lie group and D is a finite-dimensional *unitary* representation, then it is completely reducible.

Symmetries of a quantum Hamiltonian operator

Suppose that H is invariant with respect to a group of *unitary* transformations $T \in G$: $T^\dagger H T = H$. T unitary implies $[H, T] = 0$.

Take an eigenfunction $H\psi = E\psi$. Then $H(T\psi) = (HT)\psi = (TH)\psi = T(H\psi) = TE\psi = E(T\psi)$, meaning $T\psi$ is another eigenfunction for H with the same eigenvalue E.

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Quantum operators are *linear* and their eigenfunctions span a Hilbert space. Suppose the eigenfunctions with identical eigenvalue E span a d-dimensional space with basis $\{\psi^1, \ldots, \psi^d\}$.

Linearity now tells us that for each a, $T\psi^a = \sum_b t^{ab}\psi^b$.

The coefficients t^{ab} form a d-dimensional matrix representation for G, with the vector space having basis $\{\psi^1, \ldots, \psi^d\}$.
On this subspace, H acts as a multiple of the identity matrix I_d.

Schur's lemma takes many forms depending on context.

Lie group version

Let $D : G \to GL(n, \mathbb{C})$ be an irreducible representation of a matrix Lie group G. Suppose we have $a \in G$ such that $aga^{-1} = g$ for all $g \in G$. Then $D(a) = \lambda I_n$ for some $\lambda \in \mathbb{C}$.

Lie algebra version

Let $D' : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ be an irreducible representation of Lie algebra $\mathfrak{g}$. Suppose $A \in \mathfrak{gl}(n, \mathbb{C})$, that matrices A and $D'(T)$ are given with respect to the same basis for $\mathbb{C}^n$, and that $AD'(T) = D'(T)A$. Then $A = \lambda I_n$ for some $\lambda \in \mathbb{C}$.

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This points to the connection physicists exploit between a Hamiltonian operator H, its symmetry group, irreducible representations of that group, and the eigenfunctions for H.

Irreducible reps for $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$

Constructing the irreducible representations for $\mathfrak{su}(2)_{\mathbb{C}}$ follows the same procedure as finding the eigenvalues and their multiplicity for the quantum orbital angular momentum operators.

1. The generators and commutators are J^a, $[J^a, J^b] = i\epsilon^{abc}J^c$, with $a, b, c \in \{x, y, z\}$.
2. Define $C = (J^x)^2 + (J^y)^2 + (J^z)^2$ as a *Casimir element*, and $J^{\pm} = J^x \pm iJ^y$.
3. Assume $D_n : \mathfrak{su}(2)_{\mathbb{C}} \to \mathfrak{gl}(n, \mathbb{C})$ is irreducible and choose a basis for $\mathbb{C}^n$ to be the eigenvectors of J^z. C commutes with all J^a so $C = \lambda I_n$ for some λ that depends on the dimension n.
4. Use the raising and lowering operators to find that the eigenvalues of J^z must be $j, j-1, \ldots, -j+1, -j$, that $\lambda = j(j+1)$ and that $j = (n-1)/2 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.
5. We can use this information to write out the n-dimensional matrices for J^a in full for any dimension n.

Irreducible reps for $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$

$j = 0, n = 1$

This is the trivial representation. $J^x = J^y = J^z = 0$.

$j = \frac{1}{2}, n = 2$

This is the standard $\mathfrak{su}(2)$ representation in terms of the Pauli matrices. $J^a = \frac{1}{2}\sigma_a$.

$j = 1, n = 3$

This is equivalent to the standard representation for $\mathfrak{so}(3)$, but with a basis (Cartesian not “spherical”!) that makes J^z diagonal.

$$J^x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad J^y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Representations of $SU(2)$

- Since $SU(2)$ is simply connected we know representations for it are in one-to-one correspondence with those of $\mathfrak{su}(2)_{\mathbb{C}}$.
- Since $SU(2)$ is compact we know all its finite-dimensional representations are completely reducible to a direct sum of irreducible ones. This also holds for its (complexified) Lie algebra.
- [Theorem] Any two irreducible representations of $\mathfrak{su}(2)_{\mathbb{C}}$ of the same dimensions are equivalent.
- It follows that any representation of $SU(2)$ is equivalent to the direct sum of some combination of irreducible representations constructed as described on the previous slide.

Representations of $SO(3)$

- $SO(3)$ is NOT simply connected and only the $\mathfrak{su}(2)_{\mathbb{C}}$ representations with integer $j = 0, 1, 2, \ldots$ (odd dimensional reps) are true representations of $SO(3)$.
- $SO(3)$ is compact so we still have that all its finite-dimensional representations are completely reducible to a direct sum of irreducible ones.
- $\langle$Show that the $j = 1/2$ representation of $\mathfrak{su}(2)_{\mathbb{C}}$ is not a representation of $SO(3)$. $\rangle$
- $\langle$Show that the $j = 1$ representation of $\mathfrak{su}(2)_{\mathbb{C}}$ is not a *faithful* representation of $SU(2)$.$\rangle$

Representations of the Lorentz group

Recall that the complexified Lorentz Lie algebra $\mathfrak{so}^+(1,3)_\mathbb{C}$ splits into the direct sum $\mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C} = \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$.
There are six generators $N^a_\pm = \frac{1}{2}(J^a \pm iK^a)$ with commutation relations

$$[N^a_+, N^b_+] = i\epsilon^{abc}N^c_+, \quad [N^a_-, N^b_-] = i\epsilon^{abc}N^c_-, \quad [N^a_+, N^b_-] = 0$$

Every $X \in \mathfrak{so}^+(1,3)_\mathbb{C}$ can be written uniquely as $X = X_+ + X_-$ with $X_\pm = t^a N^a_\pm$. The associated Lie group* elements satisfy

$$e^{iX} = e^{iX_+ + iX_-} = (e^{iX_+})(e^{iX_-}) \quad \text{because } [N^a_+, N^b_-] = 0.$$

If X_+ and X_- did not commute, we would have to invoke the Baker-Campbell-Hausdorff formula here.

* i.e., the simply connected covering group which happens to be isomorphic to $SL(2,\mathbb{C})$.

We want to combine two representations for $\mathfrak{su}(2)_\mathbb{C}$ into one for $\mathfrak{so}^+(1,3)_\mathbb{C}$. Even though the algebras are related by a direct sum, the combination of representations is achieved using the *tensor product* of vector spaces.
A tensor product representation $D_m \otimes D_n$ for the group acts on the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$ of dimension mn as

$$(D_m \otimes D_n)(e^{iX})(u \otimes v) = e^{iD'_m(X_+)}(u) \otimes e^{iD'_n(X_-)}(v)$$

At the Lie algebra level this looks like a product rule:

$$\begin{aligned}(D_m \otimes D_n)'(X)(u \otimes v) &= (D'_m(X_+) \otimes I_n)(u \otimes v) + (I_m \otimes D'_n(X_-))(u \otimes v) \\ &= D'_m(X_+)(u) \otimes v + u \otimes D'_n(X_-)(v)\end{aligned}$$

For the Lorentz group we find that

$j = 0,\ j = 0$

This is again the trivial representation. The vector space of the representation consists of *scalars*.

$j = \frac{1}{2},\ j = 0$

$(D_2 \otimes D_1)'(X) = D_2'(X_+) \otimes I_1 + (I_2 \otimes D_1'(X_-)) \simeq D_2'(X_+)$. This becomes the *left-chiral spinor* representation.

$j = 0,\ j = \frac{1}{2}$

$(D_1 \otimes D_2)'(X) = D_1'(X_+) \otimes I_2 + (I_1 \otimes D_2'(X_-)) \simeq D_2'(X_-)$. This becomes the *right-chiral spinor* representation.

$j = \frac{1}{2}$, $j = \frac{1}{2}$

$(D_2 \otimes D_2)'(X) = D_2'(X_+) \otimes I_2 + (I_2 \otimes D_2'(X_-))$. The vector space is $\mathbb{C}^2 \otimes \mathbb{C}^2$ but this group representation acts in a way that is isomorphic to the standard 4-vector representation.

A reducible representation

The *Dirac spinor* representation is the direct sum of the left and right-chiral spinor representations:

$$D'_{\mathcal{D}}(X) = (D_2 \otimes D_1)'(X) \oplus (D_1 \otimes D_2)'(X) \simeq D_2'(X_+) \oplus D_2'(X_-)$$

These are just the simplest low-dimensional representations. Many more also have relevance in physical contexts.

- Given two irreducible representations D_m, D_n for a Lie group G, we use a tensor product to obtain an mn-dimensional representation $D_{mn}(g) = D_m(g) \otimes D_n(g)$.
- At the Lie algebra level we have product rule behaviour again with $D'_{mn}(X) = D'_m(X) \otimes I_n + I_m \otimes D'_n(X)$.
- This new representation will, in general, be *reducible*, and if G is compact, or D is unitary, we know that it is *completely reducible* and would like to find its irreducible parts.
- This procedure is “finding the Clebsch-Gordan coefficients" or “multiplying ladders”. It amounts to finding dimensions of the distinct invariant subspaces $V_{n_r} \subset \mathbb{C}^{mn}$ with $\sum n_r = mn$.

Clebsch-Gordan for $SU(2)$

Given two irreducible representations for $SU(2)$ with $j = (m-1)/2$ and $k = (n-1)/2$, assume $j \geq k$. The tensor product space for the representation $D_m \otimes D_n$ decomposes as

$$\mathbb{C}^m \otimes \mathbb{C}^n \sim \mathbb{C}^{mn} = V_{j+k} \oplus V_{j+k-1} \oplus \cdots \oplus V_{j-k}$$

where the dimension of $V_{n_r} = 2n_r + 1$.
The representation on each V_{n_r} is the unique irreducible representation for $SU(2)$ of that dimension.

⟨check the vector space dimensions for the decomposition add up appropriately for some choice of j, k.⟩

A brief outline of representations for $SU(3)$

$SU(3)$ is the group of unitary 3×3 matrices with $\det(U) = 1$. It is compact, connected and simply connected, with an 8-dimensional manifold as its parameter space. The Gell-Mann matrices given in Lecture 2 are one representation for its Lie algebra.

Properties of $\mathfrak{su}(3)_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$ exemplify those of a large class called *complex semisimple Lie algebras* which have been completely characterised.

To explain requires even more terminology...

- a subspace $V \subset \mathfrak{g}$ is an *ideal* if $[X, Y] \in V$ for all $X \in \mathfrak{g}$ and $Y \in V$.
- a Lie algebra $\mathfrak{g}$ is *simple* if $\dim \mathfrak{g} \geq 2$ and it has no non-trivial ideals.
- a *semisimple Lie algebra* is the direct sum of simple Lie algebras.

adjoint representation

Given a basis $\{T^1, \ldots, T^d\}$ for a d-dimensional Lie algebra and the structure constants $[T^a, T^b] = if^{abc}T^c$, the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(d, \mathbb{C})$ is defined by $\mathrm{ad}(T^a) = (x_{bc}) = (-if^{abc})$

A brief outline of representations for $SU(3)$

The process of finding the irreducible representations uses the following concepts

1. A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ defined so that $\mathfrak{h}$ is a maximal abelian subalgebra such that ad restricted to $\mathfrak{h}$ is completely reducible. This means the elements of $\mathfrak{h}$ can be *simultaneously diagonalised*. Their eigenvectors are used as the basis for the representation.
2. (Theorem) Any two Cartan subalgebras are isomorphic via an automorphism of $\mathfrak{g}$.
3. The *rank* of $\mathfrak{g}$ is the dimension of a Cartan subalgebra.

   For $\mathfrak{su}(2)$, the Cartan subalgebra is spanned by J^z and is of rank 1. For $\mathfrak{su}(3)$, it is T^3 and T^8, the elements with non-zero entries on the diagonal, so $\mathfrak{su}(3)$ has rank 2.

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4. A *root vector* is an eigenvector for the adjoint representation of the Cartan subalgebra and the eigenvalues are the *roots*. These act as the raising and lowering operators did to map the Cartan eigenvectors amongst themselves *in any representation*.

The process of finding the irreducible representations uses the following concepts

5. In any representation of $\mathfrak{g}$, a set of simultaneous eigenvectors and eigenvalues for the matrices $D'(H),\ H \in \mathfrak{h}$ are called *weights* and *weight vectors*.
6. (Theorem) Each irreducible representation of $\mathfrak{g}$ has a unique "highest" weight and any two irreducible reps with the same highest weight are equivalent.

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