Group Theory:

a brief introduction to some elements thereof

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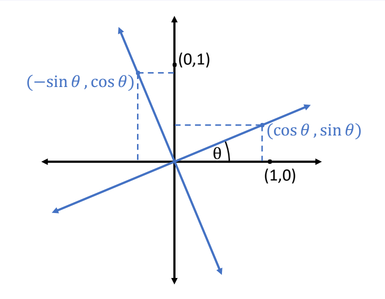
Lectures for the Canberra International Physics Summer School "Fields and Particles"

January 2023

Lecture 2: Lie algebras as linear approximation to Lie groups

Anti-clockwise rotation of the plane about the origin by an angle θ is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



We can break this down into many tiny rotations:

$$R(\theta) = \left[R(\tfrac{\theta}{N})\right]^N = \left[\begin{pmatrix} \cos\frac{\theta}{N} & -\sin\frac{\theta}{N} \\ \sin\frac{\theta}{N} & \cos\frac{\theta}{N} \end{pmatrix}\right]^N \simeq \left[\begin{pmatrix} 1 & -\frac{\theta}{N} \\ \frac{\theta}{N} & 1 \end{pmatrix}\right]^N$$

$$= \left[I + \tfrac{\theta}{N}X\right]^N \to e^{\theta X} \text{ as } N \to \infty.$$

$$\text{with } X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left.\frac{d\,R(\theta)}{d\theta}\right|_{\theta=0}$$

$SO(2)$: THE CANONICAL EXAMPLE

We have $R(\theta) = e^{\theta X}$ with $X = \left.\frac{d\,R(\theta)}{d\theta}\right|_{\theta=0}$

This is even easier to express when working with $U(1)$

$$R(\theta) = e^{i\theta} \quad \text{with } i = \left.\frac{d\,R(\theta)}{d\theta}\right|_{\theta=0}$$

⟨compute the matrix products $X^2, X^3, X^4 \ldots$ with $i^2, i^3, i^4, \ldots$ what do you notice?⟩

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Matrix Lie group generators

Suppose $g(t)$ is a continuous path of matrices in G with $g(0) = I$. Then there is a *generator*, X, defined by

$$X = \left.\frac{dg(t)}{dt}\right|_{t=0} \quad \text{and } g(t) = e^{tX} \text{ for elements along this path.}$$

The collection of all possible generators forms the *Lie algebra* $\mathfrak{g}$.

Algebraic definition of Lie algebra

A *real coefficient Lie algebra* $\mathfrak{g}$ is an n-dimensional vector space with an operation called *Lie product*, or *Lie bracket* written $[a, b]$ satisfying

1. *Closure.* If $a, b \in \mathfrak{g}$ then $[a, b] \in \mathfrak{g}$.
2. *Linearity.* If $a, b, c \in \mathfrak{g}$, and $\alpha, \beta \in \mathbb{R}$, then $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]$.
3. *Anti-symmetry.* $[a, b] = -[b, a]$ for all $a, b \in \mathfrak{g}$.
4. *Jacobi's identity.* For $a, b, c \in \mathfrak{g}$, $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

If the elements are square matrices, the Lie product is the commutator: $[a, b] = ab - ba$.

Where did the commutator come from?

Origin of the matrix commutator

Multiplication in a matrix Lie group G requires that $e^X e^Y = g \in G$, with $g = e^Z$ for some other generator $Z \in \mathfrak{g}$.

»!!! Since X, Y are matrices $e^X e^Y \neq e^{X+Y}$!!!«

Rather, we have

The Baker-Campbell-Hausdorff formula

$$e^X e^Y = \left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{Y^n}{n!}\right)$$
$$= \exp\left[X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots\right]$$

So we see that Z is expressed as a sum of matrix commutators, and this the origin of the Lie bracket.

$\langle$alt., suppose $a = (I + A)$, $b = (I + B)$ are close to $I \in G$, show $aba^{-1} \simeq b + [A, B]$. $\rangle$

Lie algebra of $SO(3)$

For a fixed direction $\vec{n}$, the rotation $R(\vec{n}, \theta)$ is a continuous path of group elements with $R(\vec{n}, 0) = I$. Set $X = \left.\frac{dR(\vec{n},\theta)}{d\theta}\right|_{\theta=0}$, so that $R(\vec{n}, \theta) = e^{\theta X}$.

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The defining property of matrices in $SO(3)$ is $R^T = R^{-1}$. For elements along the path generated by X, this implies $e^{\theta X^T} = e^{-\theta X}$ for all θ, so that $X^T = -X$.

A generator for $SO(3)$ must have the form $X = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_1 & 0 \end{pmatrix}$ and we see that the vector space for the Lie algebra $\mathfrak{so}(3)$ is 3×3 *antisymmetric matrices*.

$\langle$compute the product of two matrices $X, Y \in \mathfrak{so}(3)$. Is XY antisymmetric?$\rangle$

$\langle$show that $[X, Y] = XY - YX$ is antisymmetric if X, Y are $\rangle$

Lie algebra of $SO(3)$ (physics version)

Before going any further, we must switch to physics notation, and introduce a factor of i into the matrix exponential definition. This converts the generators of $SO(3)$ from being antisymmetric to being *hermitean*, and more closely related to quantum observables.

We now have $R(\vec{n}, \theta) = e^{i\theta J}$ with $J = -iX$ ⟨show that $J^\dagger = J$.⟩
The standard physics basis for the Lie algebra $\mathfrak{so}(3)$ is then

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subscripts x, y, z refer to rotation about x,y,z-axes respectively.

Direct calculation shows $[J_x, J_y] = iJ_z,\ [J_y, J_z] = iJ_x,\ [J_z, J_x] = iJ_y$.

Basis, structure constants

The vector space structure of the Lie algebra greatly simplifies the amount of information required to describe it.

Let $\{T^1, \ldots, T^m\}$ be a basis for an m-dimensional real-coefficient Lie algebra. Every $T \in \mathfrak{g}$ is uniquely written as $T = \sum_{a=1}^{m} \theta^a T^a$, with $\theta^a \in \mathbb{R}$.

The commutators are $[T^a, T^b] = i \sum_{c=1}^{m} f^{abc} T^c$, and the numbers f^{abc} are called the *stucture constants*.

- From now on we will omit the summation symbol and assume repeated indices are to be summed over.
- The factor of i is required above as we are using the physics definition $g = e^{i\theta^a T^a}$

Lie group and algebra of $SU(2)$

A matrix $U \in SU(2)$ has $U^\dagger U = 1$ and $\det U = 1$.
Suppose $U = e^{iH}$. $U^\dagger U = 1$ implies $(e^{-iH^\dagger})(e^{iH}) = I = e^0 \implies H^\dagger = H$.
Properties of the matrix exponential show that $\det U = 1 \implies \operatorname{tr} H = 0$.
So $\mathfrak{su}(2)$ contains 2×2 traceless hermitean matrices:

$$H = \begin{pmatrix} w & u - iv \\ u + iv & -w \end{pmatrix}$$

Recall that the manifold for $SU(2)$ is the 3-sphere so we expected its Lie algebra to have a basis of three generators. These are conventionally written in terms of the Pauli matrices $s_k = \frac{1}{2}\sigma_k$

$$s_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

The commutators are

$$[s_1, s_2] = is_3 \;\; [s_2, s_3] = is_1 \;\; [s_3, s_1] = is_2.$$

$\mathfrak{so}(3) = \mathfrak{su}(2)$

If you have a sharp short-term memory, you will have noticed that both $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are 3-dimensional and the structure constants for our choice of basis are identical. 🤯

This illustrates a deep theorem from Lie theory:

Covering group theorem

If G is a connected matrix Lie group, then a (connected and simply connected) universal covering group $\tilde{G}$ of G exists and is unique up to isomorphism.
If $\tilde{G}$ is also a matrix Lie group, then the Lie algebras of G and $\tilde{G}$ are isomorphic.

The manifold for $SU(2)$ is the 3-sphere; S^3 is simply-connected and double-covers $\mathbb{R}P^3$, the manifold of $SO(3)$.

$\mathfrak{so}(3)$ = QUANTUM ORBITAL ANGULAR MOMENTUM

If you have a sharp long-term memory, you'll also recall that components of the orbital angular momentum operator are

$$L_x = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}), \quad L_y = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}), \quad L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

and $[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y.$ The *algebraic structure* of orbital angular momentum is the same as $\mathfrak{so}(3)$.

We also have that $L^2 = L_x^2 + L_y^2 + L_z^2$, and $[L^2, L_a] = 0$.

Perhaps you also remember that two *ladder operators* $L_\pm = L_x \pm iL_y$ allow us to find the eigenvalues for L^2, and their multiplicity.

Exactly the same procedure for $\mathfrak{so}(3) = \mathfrak{su}(2)$ gives us the *irreducible representations* of the Lie algebra.
L^2 is called a *Casimir element*. The eigenvalue is a function of its multiplicity, d, which is the dimension of the irreducible representation.

The ladder operators $L_\pm = L_x \pm iL_y$ introduced a *complex coefficient* into the linear combination of algebra elements.

Complexification

Given a real-coefficient Lie algebra, $\mathfrak{g}$, we define a complex-coefficient Lie algebra $\mathfrak{g}_\mathbb{C}$ by using the same basis elements but allowing complex coefficients.

Example: $\mathfrak{su}(2)_\mathbb{C} = \mathfrak{sl}(2;\mathbb{C}) = \mathfrak{sl}(2;\mathbb{R})_\mathbb{C}$

BUT: $\mathfrak{su}(2) \neq \mathfrak{sl}(2;\mathbb{R})$. They are two distinct *real forms* of the complex Lie algebra $\mathfrak{sl}(2;\mathbb{C})$.

Extending the Lie algebra coefficient group to the complex field greatly assists with finding representations.

The Lorentz group and its Lie algebra

The Lorentz group preserves Minkowski space-time intervals:
$\Lambda \in O(1,3)$ satisfies $\Lambda^T g_M \Lambda = g_M$, where $g_M = \mathrm{diag}\{1,-1,-1,-1\}$.
This condition implies $\det\Lambda = \pm 1$, and $\Lambda_{00}^2 = 1 + \Lambda_{10}^2 + \Lambda_{20}^2 + \Lambda_{30}^2$

$O(1,3)$ has four components. The part connected to I is called $SO^+(1,3)$. $\Lambda \in SO^+(1,3)$ has $\det\Lambda = 1$, $\Lambda_{00} \geq 1$.

The Lie algebra $\mathfrak{so}^+(1,3)$ has six generators: $\Lambda = e^{i(\theta^a J^a + \nu^b K^b)}$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \text{ etc.} \quad K^1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.}$$

with $[J^a, J^b] = i\epsilon^{abc}J^c, \quad [J^a, K^b] = i\epsilon^{abc}K^c, \quad [K^a, K^b] = -i\epsilon^{abc}J^c.$

We now move to the complexified version of the Lie algebra and define six new generators: $N^a_\pm = \frac{1}{2}(J^a \pm iK^a)$.
These have commutation relations

$$[N^a_+, N^b_+] = i\epsilon^{abc}N^c_+, \quad [N^a_-, N^b_-] = i\epsilon^{abc}N^c_-, \quad [N^a_+, N^b_-] = 0$$

This shows us that

$$\mathfrak{so}^+(1,3)_\mathbb{C} = \mathfrak{su}(2)_\mathbb{C} \oplus \mathfrak{su}(2)_\mathbb{C} = \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C}).$$

The symbol ϵ^{abc} is the permutation symbol or Levi-Civita symbol

$$\epsilon_{abc} = \begin{cases} 1 & \text{if } abc = 123, 231, 312 \\ -1 & \text{if } abc = 132, 213, 321 \\ 0 & \text{otherwise. i.e., repeated indices} \end{cases}$$

$SU(3)$ and its Lie algebra

$SU(3)$ is a local symmetry of the Lagrangian for three fermion fields. It is a simply-connected and compact Lie group.
$U \in SU(3)$ satisfies $U^\dagger U = I$ and $\det U = 1$. The Lie algebra defined by $U = e^{itH}$ is 8-dimensional with $H^\dagger = H$ and $\operatorname{tr} H = 0$. A basis is given by the *Gell-Mann matrices* $T^a = \frac{1}{2}\lambda^a$ with

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$SU(3)$ and its Lie algebra

The structure constants for $\mathfrak{su}(3)$ are defined by $[T^a, T^b] = if^{abc}T^c$, where $a, b, c \in \{1, \ldots, 8\}$ and

$$f^{123} = 1;\ f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2};\ f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

all permutations of the above indices take $\pm$ values as appropriate (e.g., $f^{132} = -f^{123} = -1$) and indices not defined are 0 (e.g., $f^{134} = 0$).

Fundamental questions (again)

Q: How to label elements of uncountably infinite groups?

- We parametrise them, and determine the allowed space of parameters.
- (Matrix) Lie groups are a special type of continuous group, where the parameter space is a differentiable manifold and the product and inverse are continuous maps.
- A (finite-dimensional matrix) Lie algebra is obtained as the linearisation of a matrix Lie group near its identity element.
- The exponential map takes us from the algebra to the group $g = e^{iT}$ but is only a homeomorphism (1-1 and onto) in a neighbourhood of the identity.

Fundamental questions (again)

Q: How do we determine when two groups are "the same"?

- A complete answer is not possible in general.
- Two Lie groups are "the same" if there is a homeomorphism of their manifolds that preserves the group product: $\phi : G \to H$ with $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.
- Two Lie algebras are "the same" if there is a linear mapping $\varphi : \mathfrak{g} \to \mathfrak{h}$ with $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ that is one-to-one and onto (an isomorphism).
- Two different Lie groups can have isomorphic Lie algebras.

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