Group Theory:

a brief introduction to some elements thereof

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Lectures for the Canberra International Physics Summer School "Fields and Particles"

January 2023

Outline

1. Groups encountered in physics
2. Lie algebras as linear approximation to Lie groups
3. Representations of Lie groups and algebras
4. Symmetries of things: discrete groups in condensed matter

References from more physics-y to more maths-y

- A. Zee "Group theory in a nutshell for physicists" Princeton University Press (2016)
- J. Schwichtenberg "Physics from symmetry" 2nd ed. Springer (2018)
- J.F. Cornwell "Group Theory in Physics : An Introduction" Elsevier (1997)
- B.C. Hall "An Elementary Introduction to Groups and Representations" arXiv:math-ph/0005032v1 (2000) expanded and published as Springer GTM vol.222, (2003)

Lecture 1: Groups encountered in physics

A group is the mathematical formulation of symmetry

A symmetry is a transformation that leaves something unchanged: $T : M \to M$;
e.g., $T(x, y) = (-x, y)$ is a reflection in the y-axis and a symmetry of the unit disk.

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Newton's equation of motion $\vec{F} = m\vec{a}$ is said to be *covariant*, while kinetic energy $T = \frac{1}{2}m(\vec{v} \cdot \vec{v})$ is *invariant* under a rotation of coordinate axes.

Example

A Lagrangian invariant under Euclidean isometries

$$\mathcal{L}\left(\frac{d\vec{q}}{dt}, \vec{q}\right) = \mathcal{T}\left(\frac{d\vec{q}}{dt}\right) - \mathcal{V}(\vec{q}) \qquad \mathcal{L} : (M, TM) \to \mathbb{R}$$

Suppose $M = \mathbb{R}^3$ and that $\mathcal{L}$ is invariant under isometries $A : \mathbb{R}^3 \to \mathbb{R}^3$ with $\|A(\vec{q}_1 - \vec{q}_2)\| = \|\vec{q}_1 - \vec{q}_2\|$. Then

$$\mathcal{L}\left(\frac{d\vec{q}}{dt}, \vec{q}\right) = \mathcal{L}\left(\frac{dA\vec{q}}{dt}, A\vec{q}\right)$$

and this implies $\mathcal{L}$ can be a function of $\|\vec{q}\|$ and $\|\frac{d\vec{q}}{dt}\|$ only.

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This argument extends to Lagrangian densities of quantum field theory. If a field theory is invariant under a group of symmetries, this constrains the functional form of the Lagrangian.

Definition of a group

$(G, *)$ is a set of elements $a \in G$ with binary operation $a * b$ such that the following hold

1. *Closure*. If $a, b, \in G$, then $a * b \in G$.
2. *Associativity*. $(a * b) * c = a * (b * c)$.
3. *Identity element*. There is an $I \in G$, such that $a * I = I * a = a$ for all $a \in G$.
4. *Inverses*. If $a \in G$, then there is $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = I$.

If the group operation also satisfies $a * b = b * a$, the operation is called *commutative* and the group is called *Abelian*.

Definition of a group

$(G, *)$ has *closure, associativity, identity element, inverses*

Examples

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- Complex numbers with multiplication: $(\mathbb{C} \setminus 0, *)$. What is the identity? an inverse? Is this group Abelian?

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- Complex numbers with multiplication: $(\mathbb{C} \setminus 0, *)$. What is the identity? an inverse? Is this group Abelian?
- Square, $n \times n$, invertible matrices with entries in $\mathbb{R}$ (or $\mathbb{C}$) and matrix multiplication as the binary operation form the *general linear group* $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$).

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- Isometries of $\mathbb{R}^n$ with composition form the *Euclidean group*.
- Transformations of Minkowski spacetime that preserve $(c\Delta t)^2 - (\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2$ form the *Poincaré group*.

Subgroups

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Examples

- The *trivial group* (or subgroup) has just the identity element.
- Symmetry groups of polyhedra are finite subgroups of $GL(3, \mathbb{R})$.
- The n-dimensional lattice group $(\mathbb{Z}^n, +)$ is a countably infinite subgroup of $(\mathbb{R}^n, +)$.
- Euclidean isometries that fix the origin are an uncountably infinite subgroup of $GL(n, \mathbb{R})$ called $O(n)$.

Fundamental questions

Q: How to label elements of uncountably infinite groups?

We parametrise them, and determine the allowed space of parameters.

Q: How do we determine when two groups are “the same”?

A complete answer is not possible in general, but we will learn some methods of attack in the next few lectures.

Definition of a Lie Group

A *Lie group* is a group, $(G, *)$ whose elements are parametrised by a differentiable manifold M, so that each $x \in M$ is in 1-1 correspondence with an element $g(x) \in G$, and

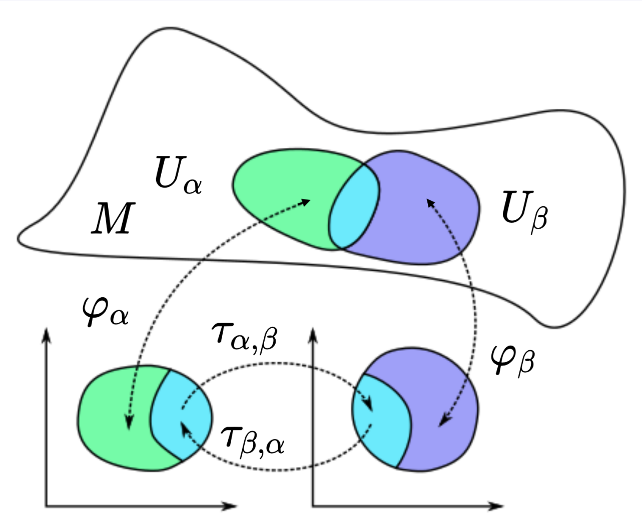
the product induces a differentiable map

Given $x, y, z \in M$, such that $g(x) * g(y) = g(z)$, the function $\phi : M \times M \to M$ defined by $\phi(x, y) = z$, is differentiable.

the inverse defines a differentiable map

Given $x, y \in M$ such that $g(x) * g(y) = g(y) * g(x) = I$, the function $\nu : M \to M$ defined by $\nu(x) = y$ is differentiable.

(Definition of a Manifold)



Lie group	Compact	Connected	Dimension	Definition
$GL(n,\mathbb{R})$	N	N	n^2	$\det A \neq 0$
$SL(n,\mathbb{R})$	N	Y	n^2-1	$\det A = 1$
$O(n)$	Y	N	$n(n-1)/2$	$A^TA = I$
$SO(n)$	Y	Y	$n(n-1)/2$	$A^TA = I$, $\det A = 1$
$O(1,3)$	Y	N	6	$\Lambda^T g \Lambda = g$
$SO^+(1,3)$	Y	Y	6	$\det \Lambda = 1$, $\Lambda^0_0 \geq 1$
$U(n)$	Y	Y	n^2	$A^\dagger A = I$
$SU(n)$	Y	Y	n^2-1	$A^\dagger A = I$, $\det A = 1$

g is the Minkowski metric tensor $g_{00} = 1, g_{11} = g_{22} = g_{33} = -1$, $g_{ij} = 0$.

Of the above groups, only $SU(n)$ are simply connected.

Common Matrix Lie groups

- The manifold for $U(1)$ and for $SO(2)$ is the unit circle S^1.
  $$U(1) = \{z \in \mathbb{C} \mid z^\dagger z = 1\} = \{e^{i\theta},\ \theta \in [0, 2\pi)\}.$$
- The manifold for $SU(2)$ is the 3-sphere S^3.
- The manifold for $SO(3)$ is $\mathbb{R}P^3$ (the 3-sphere with antipodal points identified).
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Asides:

The group structure on S^1 comes from multiplication of unit complex numbers.

The group structure on S^3 comes from multiplication of unit *quaternions*.

There is no Lie group associated with the 2-sphere, S^2.

SO(3): The Rotation Group

Algebraically

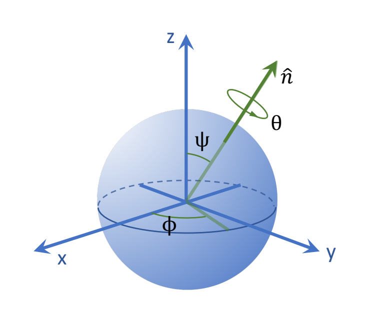
$A \in SO(3)$ means A is a 3×3 real matrix with $A^T A = I$ and $\det A = 1$. So we have 9 variables with 6 constraints ⟨why?⟩, implying that $SO(3)$ should be a 3-dimensional manifold.

Geometrically

The isometry $R(\vec{n}, \theta) \in SO(3)$, acts on $\mathbb{R}^3$ as a rotation by angle θ about a line through the origin in direction $\vec{n}$.
We use a right hand rule to specify that the rotation is anti-clockwise when your thumb points along $\vec{n}$.

If $\vec{n}$ points along the z-axis,

$$R(\vec{z}, \theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



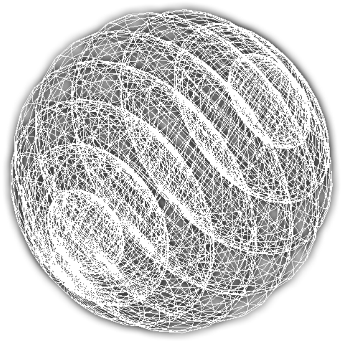
Local versus Global structure

The manifold for $SO(3)$ is NOT $S^2 \times S^1$.
This is because (as matrices) $R(\vec{n}, \theta) = R(-\vec{n}, -\theta)$

Note also that $R(\vec{n}, 0) = I$ for any $\vec{n}$.
$\langle$What happens when $\theta = \pi$? $\rangle$

The fact that the manifold for $SO(3)$ is $\mathbb{R}P^3$ is most easily proved by mapping vectors into a subspace of the quaternions.

We will see later that $SO(3)$ and $SU(2)$ are closely related.



A few more definitions

Conjugacy

Two group elements a, b are said to be *conjugate* if there is a third element so that $b = gag^{-1}$. The *conjugacy class* of $a \in G$ is the set of elements $Cl(a) = \{gag^{-1} \mid g \in G\}$

If G is a matrix group, then all elements of a conjugacy class have the same rank, determinant, trace, eigenvalues and their geometric multiplicities (but not the eigenvectors!). This is succinctly summarised by the phrase

"Character is a function of class".

Example: In $SO(3)$, $Cl(R(\vec{n}, \theta)) = \{R(\vec{u}, \theta) \mid \vec{u} \in S^2\}$.

A few more definitions

Normal subgroup

A subgroup $N \subset G$ is *normal* if for $n \in N$, and all $g \in G$, $gng^{-1} \in N$. G and $\{I\}$ are referred to as the *trivial normal subgroups* of G.

A *simple group* is one that has no non-trivial normal subgroups.

In an abelian group, all subgroups are normal.

Example: $SO(3)$ is a simple group.

Example: $O(3)$ has a non-trivial normal subgroup $N = \{I, -I\}$. $SO(3)$ is also a normal subgroup of $O(3)$.

⟨show that translations form a normal subgroup of the Euclidean group⟩

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