Lecture 6: Divergences, renormalization, renormalization group, regularization, QED beta function, running mass and coupling

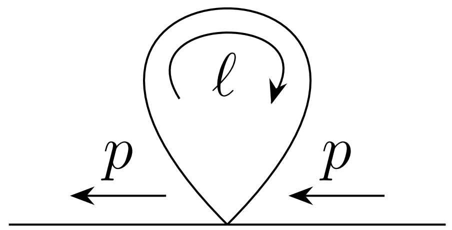
Loops lead to divergences

(See Chapter 8, Sec 8.6.1)

Example: Consider the one-loop correction to the scalar propagator on $\lambda\phi^4$ theory.

Consider the single loop contribution to the scalar 1PI self-energy $\Pi(p^2)$ in $(\lambda/4!)\phi^4$ theory in Eq. (7.5.49). Using the Feynman rules in Sec. 7.6.1 we have $-i\lambda$ for the vertex from Eq. (7.6.10), a symmetry factor of $S = 2$ from a vertical flip of the loop, a scalar propagator $i/\ell^2 - m^2 + i\epsilon$ and a loop integral $\int d^4\ell/(2\pi)^4$. Define the $\mathcal{O}(\lambda^1)$ contribution to $-i\Pi(p^2)$ as $-i\Pi_1(p^2)$. With a cut-off it is

$$-i\Pi_1(p^2) =$$



$$= -i\frac{\lambda}{2}\int^{\Lambda} \frac{d^4\ell}{(2\pi)^4}\frac{i}{\ell^2 - m^2 + i\epsilon}\,, \qquad (8.6.3)$$

where the result is independent of p^2, there are no external lines on a 1PI diagram, and the subscript 1 denotes $\mathcal{O}(\lambda^1)$. Rotating to Euclidean space and using rotational invariance, $d^4\ell \to id^4\ell_E = i\pi^2\ell_E^2 d\ell_E^2$ and $k^2 \to -k_E^2$ we find

$$\begin{aligned} -i\Pi_1 &= -i\lambda\tfrac{1}{2}[\pi^2/(2\pi)^4]\int^{\Lambda^2} d\ell_E^2\,[\ell_E^2/(\ell_E^2 + m^2)] \qquad (8.6.4)\\ &= -i(\lambda/32\pi^2)\int^{\Lambda^2} d\ell_E^2\,[1 - m^2/(\ell_E^2 + m^2)] = -i(\lambda/32\pi^2)[\Lambda^2 - m^2\ln(\{\Lambda^2 + m^2\}/m^2)]\\ &= -i(\lambda/32\pi^2)[\Lambda^2 - m^2\ln(\Lambda^2/m^2) + \mathcal{O}(1/\Lambda^2)]\,. \end{aligned}$$

We see that Π_1 is both quadratically and logarithmically divergent as $\Lambda \to \infty$.

Renormalization of QED

(See Chapter 8, Sec 8.5)

BPHZ theorem: *All divergences in a renormalizable theory are removed by the renormalization of the primitively divergent diagrams through the renormalization of fields, masses and coupling constants.*

QED is a renormalizable theory: In a renormaliaable theory like QED evaluating loop intergals does not introduce new interactions. All divergences can be absorbed into renormalization of the fields, masses and so we can define the remormalized QED action in R_ξ gauges as

$$\begin{aligned}
\tilde{S}[\bar{\psi},\psi,A_\mu] &= \int d^dx\left[\sum_{f=1}^{N_f}\bar{\psi}_0^f\left(i\partial\!\!\!/ - m_0^f - q_0^f A\!\!\!/_0\right)\psi_0^f - \tfrac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} - \tfrac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2\right] \\
&= \int d^dx\left[\sum_f\left\{Z_2^f\bar{\psi}^f(i\partial\!\!\!/ - m_0^f)\psi^f - Z_2^f\sqrt{Z_3}q_0^f\bar{\psi}^f A\!\!\!/\psi^f\right\} - Z_3\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - Z_3\tfrac{1}{2\xi_0}(\partial_\mu A^\mu)^2\right] \\
&= \int d^dx\left[\sum_f\left\{Z_2^f\bar{\psi}^f(i\partial\!\!\!/ - m_0^f)\psi^f - Z_1^f q^f\bar{\psi}^f A\!\!\!/\psi^f\right\} - Z_3\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\xi}(\partial_\mu A^\mu)^2\right] \\
&= \int d^dx\left[\sum_f\bar{\psi}^f\left(i\partial\!\!\!/ - m^f - q^f A\!\!\!/\right)\psi^f - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\xi}(\partial_\mu A^\mu)^2\right. \qquad (8.5.1) \\
&\quad \left. + \sum_f\left\{(Z_2^f - 1)\bar{\psi}^f i\partial\!\!\!/\psi^f - (Z_2^f m_0^f - m^f)\bar{\psi}^f\psi^f - (Z_1^f - 1)q^f\bar{\psi}^f A\!\!\!/\psi^f\right\} - (Z_3 - 1)\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \\
&= \int d^dx\left[\sum_f\bar{\psi}^f\left(i\partial\!\!\!/ - m^f - q^f A\!\!\!/\right)\psi^f - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\xi}(\partial_\mu A^\mu)^2\right. \\
&\quad \left. + \sum_f\left\{\bar{\psi}^f\left(i\delta_2^f\partial\!\!\!/ - \delta_m^f\right)\psi^f - \delta_1^f q^f\bar{\psi}^f A\!\!\!/\psi^f\right\} - \delta_3\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right],
\end{aligned}$$

Renormalization of QED

(See Chapter 8, Sec 8.5)

where the bare fields, masses, charges and gauge parameter are $\psi_0^f, A_0^\mu, m_0^f, q_0^f, \xi_0$ respectively and their renormalized versions are $\psi^f, A^\mu, m^f, q^f, \xi$. We have the multiplicative renormalization constants: (i) Z_2^f is the wavefunction renormalization constant for flavor f; (ii) Z_3 is the wavefunction renormalization constant for the photon; and (iii) Z_1^f is the charge renormalization constant for flavor f. Comparing the first, second and third lines of Eq. (8.5.6) we see that we have defined

$$\psi_0^f \equiv \sqrt{Z_2^f}\psi^f \; ; \quad A_0^\mu \equiv \sqrt{Z_3}A^\mu \; ; \quad q_0^f \equiv \frac{Z_1^f}{Z_2^f\sqrt{Z_3}}q^f \; ; \quad \xi_0 \equiv Z_3\xi \, . \tag{8.5.2}$$

Comparing the last two equalities we see that we have the counterterm coefficients

$$Z_2^f \equiv 1+\delta_2^f \; ; \quad Z_3 \equiv 1+\delta_3 \; ; \quad Z_2^f m_0^f \equiv m^f + \delta m^f \; ; \quad Z_1^f \equiv 1+\delta_1^f \, . \tag{8.5.3}$$

Ultraviolet regulator and renormalization point: We will need to introduce some *effective regulator to control divergences* Λ, where the removal of the regulator means $\Lambda \to \infty$. We also need to introduce some scale at which we will define the parameters of the theory. We define this scale as the renormalization point μ . Clearly the renormalization contestants are a function of Λ. In a renormalizable theory we can hold the parameters of the theory fixed at μ and as we take $\Lambda \to \infty$, all divergences are absorbed into the renormalization constants.

Since the choice of μ was arbitrary, we can change μ and the renormalized charges/coupling constants $q(\mu)$ and masses $m(\mu)$ so that the physical properties of the renormalized theory remain unchanged. The transformations that do this make up that we call the *renormalization group*.

Renormalization of QED

(See Chapter 8, Sec 8.5)

For notational simplicity we consider a single fermion flavor with bare mass m_0 and bare charge q_0. Consider an arbitrary QED bare Green's function, $G^{(n_\psi,n_\gamma)}_{\rm bare}$, involving the bare fields A_0^ν, $\bar\psi_0$ and ψ_0 with a total of n_ψ fermion operators and n_γ photon operators. Since we have $A_0^\nu = \sqrt{Z_3(\mu,\Lambda)}A^\nu$, $\bar\psi_0 = \sqrt{Z_2(\mu,\Lambda)}\bar\psi$ and $\psi_0 = \sqrt{Z_2(\mu,\Lambda)}\psi$ the bare fields are functions of Λ and the renormalized fields are functions of μ. The bare and renormalized Green's functions are related by

$$\begin{aligned}
&G^{(n_\psi,n_\gamma)}_{{\rm bare}\,\nu_1,\ldots,\nu_{n_\gamma}}(\cdots) = \langle\Omega|T\hat A_{0\nu_1}(z_1)\cdots\hat A_{0\nu_{n_\gamma}}(z_{n_\gamma})\hat\psi_0(x_1)\cdots\hat{\bar\psi}_0(x_{n_\psi})|\Omega\rangle \\
&= [Z_2(\mu,\Lambda)]^{n_\psi/2}[Z_3(\mu,\Lambda)]^{n_\gamma/2}\langle\Omega|T\hat A_{\nu_1}(z_1)\cdots\hat A_{\nu_{n_\gamma}}(z_{n_\gamma})\hat\psi(x_1)\cdots\hat{\bar\psi}(x_{n_\psi})|\Omega\rangle \\
&= [Z_2(\mu,\Lambda)]^{n_\psi/2}[Z_3(\mu,\Lambda)]^{n_\gamma/2}G^{(n_\psi,n_\gamma)}_{\nu_1,\ldots,\nu_{n_\gamma}}(\cdots)\,. \qquad (8.5.51)
\end{aligned}$$

For brevity we now suppress spacetime indices and coordinates. Note that $G^{(n_\psi,n_\gamma)}_{\rm bare}$ depends only on the bare quantities Λ, $q_0(\Lambda)$, $m_0(\Lambda)$ and $\xi_0(\Lambda)$. The renormalized Green's function $G^{(n_\psi,n_\gamma)}$ is independent of Λ since QED is renormalizable but depends on the renormalization scale μ and on $q(\mu)$, $m(\mu)$ and $\xi(\mu)$. For simplicity we work in Feynman gauge here so that $\xi_0(\Lambda) = Z_3(\mu,\Lambda)\xi(\mu) = 0$. So we can write

$$G^{(n_\psi,n_\gamma)}_{\rm bare}(\Lambda, q_0(\Lambda), m_0(\Lambda)) = [Z_2(\mu,\Lambda)]^{n_\psi/2}[Z_3(\mu,\Lambda)]^{n_\gamma/2}G^{(n_\psi,n_\gamma)}(\mu, q(\mu), m(\mu))\,.$$

Renormalizaton group equations

(See Chapter 8, Sec 8.5)

Recall that we remove the regulator, $\Lambda \to \infty$, while holding all renormalized quantities fixed at any chosen renormalization point, μ. Similarly we vary the renormalization point μ while holding all bare quantities fixed. So by construction

$$0 = \mu \frac{\partial}{\partial \mu} G_{\text{bare}}^{(n_\psi, n_\gamma)} = \mu \frac{\partial}{\partial \mu} \left([Z_2(\mu, \Lambda)]^{n_\psi/2} [Z_3(\mu, \Lambda)]^{n_\gamma/2} G^{(n_\psi, n_\gamma)}(\mu, q(\mu), m(\mu)) \right)$$
$$= Z_2^{n_\psi/2} Z_3^{n_\gamma/2} \left(\mu \frac{\partial}{\partial \mu} + \frac{n_\gamma}{2}\gamma_3 + \frac{n_\psi}{2}\gamma_2 + \beta \frac{\partial}{\partial q} + \gamma_m m \frac{\partial}{\partial m} \right) G^{(n_\psi, n_\gamma)}, \qquad (8.5.52)$$

where we have defined

$$\beta \equiv \mu \frac{\partial q}{\partial \mu}, \quad \gamma_3 \equiv \frac{\mu}{Z_3} \frac{\partial Z_3}{\partial \mu}, \quad \gamma_2 \equiv \frac{\mu}{Z_2} \frac{\partial Z_2}{\partial \mu}, \quad \gamma_m \equiv \frac{\mu}{m} \frac{\partial m}{\partial \mu}. \qquad (8.5.53)$$

We have arrived at the *Callan-Symanzik equation*, (Callan, 1970; Symanzik, 1970),

$$\left(\mu \frac{\partial}{\partial \mu} + \frac{n_\gamma}{2}\gamma_3 + \frac{n_\psi}{2}\gamma_2 + \beta \frac{\partial}{\partial q} + \gamma_m m \frac{\partial}{\partial m} \right) G^{(n_\psi, n_\gamma)} = 0. \qquad (8.5.54)$$

We refer to β as the *β-function* and to γ_m as the *anomalous mass dimension*. Equations such as the Callan-Symanzik equation are often referred to collectively as *renormalization group equations.*

Regularization methods

(See Chapter 8, Sec 8.6.1)

Momentum cut-off: A simple momentum cut-off, Λ, violates both Lorentz invariance and gauge invariance. Violations of gauge invariance remain even after $\Lambda \to \infty$. For example, we find that $q_\mu \Pi^{\mu\nu}(q)|_{\Lambda\to\infty} \neq 0$. To illustrate why this is, substitute $S(p) \to \not{p} + m/p^2 - m^2 + i\epsilon$ into Eq. (8.5.20) with a cut-off to give $q_\nu \Pi^{\mu\nu}(q) \propto \int^\Lambda d^4\ell \, \mathrm{tr}[\gamma^\mu \{S^f(\ell+q) - S^f(\ell)\}]$. We can evaluate perturbation theory integrals in Euclidean space since the Wick rotation gives no contribution from curves C_1 and C_3 for ℓ^0 in Fig. A.1 of Sec. A.4. From Eqs. (A.4.6) and (A.6.2),

$$k^2 \to -k_E^2\,, \quad d^4\ell \to i d^4\ell_E \quad \text{and} \quad d^4\ell_E = (2\pi^2)\ell_E^3 d\ell_E = \pi^2 \ell_E^2 d\ell_E^2\,. \tag{8.6.1}$$

Evaluate the trace, retain leading terms as $\ell^2 \to \infty$ after analytic continuation to Euclidean space. This leads to

$$q_\nu \Pi^{\mu\nu}(q) \sim \int^\Lambda d^4\ell \, \mathrm{tr}[\gamma^\mu \not{q} - \gamma^\mu \not{\ell}(2q\cdot\ell/\ell^2)]/\ell^2 \sim q^\mu \int^{\Lambda^2} d\ell_E^2 \sim q^\mu \Lambda^2\,, \tag{8.6.2}$$

which does not vanish as $\Lambda \to \infty$. We have not bothered to indicate the rotation of q^μ and γ^μ to Euclidean space and back. This result remains true if we attempt to use some smooth cut-off such as a Gaussian function since any kind of cut-off destroys the momentum-translational invariance of the loop integral. So momentum cut-offs are not appropriate regulators in gauge theories.

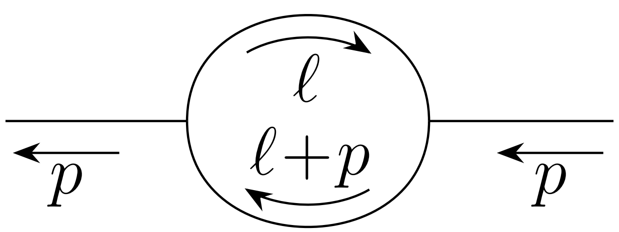
Regularization methods

(See Chapter 8, Sec 8.6.1)

Pauli-Villars regularization: The essence of Paul-Villars regularization (Pauli and Villars, 1949) is to subtract from any loop integral the same loop integral with a much larger mass M in the propagators. This suppresses the loop integral at large loop momenta, $\ell^2 \gg M^2$, where masses and external momenta are unimportant.

For example, at one-loop in $(\kappa/3!)\phi^3$ theory with mass m we have an $\mathcal{O}(\kappa^2)$ 1PI contribution to the ϕ self-energy, $-i\Pi(p^2)$, given by

$$-i\Pi_2(p^2) =$$



$$\equiv (-i\kappa)^2 \tfrac{1}{2} I(p^2)\,, \quad \text{where} \qquad (8.6.5)$$

$$I(p^2) \equiv \int \frac{d^4\ell}{(2\pi)^4} \left[\frac{i}{\ell^2 - m^2 + i\epsilon} \frac{i}{(p+\ell)^2 - m^2 + i\epsilon} - \frac{i}{\ell^2 - M^2 + i\epsilon} \frac{i}{(p+\ell)^2 - M^2 + i\epsilon} \right].$$

There is one $(-i\kappa)$ for each vertex, a symmetry factor $S = 2$ for 2! ways to exchange lines in the loop and no external propagators. Use $\ell^0 \to i\ell_4^E$, $\ell^2 \to -\ell_E^2$, and $d^4\ell_E = 2\pi^2 \ell_E^2 d\ell_E^2 = \pi^2 \ell_E^2 d\ell_E^2$. At large ℓ_E^2 we have $I(0) = (i/16\pi^2)(-1)^2 \int d\ell_E^2\, \ell^2 [\mathcal{O}(1/\ell_E^4) - \mathcal{O}(1/\ell_E^4)]$ and so the large ℓ_E^2 contribution to I is $\int d\ell_E^2\, \mathcal{O}(1/\ell_E^4)$ and so converges.

Regularization methods

(See Chapter 8, Sec 8.6.1)

Pauli-Villars regularization was widely used in the early days of quantum field theory. For an application to QED see for example Bjorken and Drell (1965), Das (2008) and Schwartz (2013). It preserves the momentum translational invariance of momentum integrations, while also being an intuitively satisfying means of controlling large-momentum behavior. However, there are a number of shortcomings that limit its modern use. Firstly, for diagrams with multiple loops one ghost per particle is insufficient and additional ghost particles are required. While it can be useful in abelian gauge theories where a massive Proca vector boson coupled to conserved currents is consistent, the method breaks down in the case of nonabelian gauge theories where massive vector bosons can not be consistently described. The method also fails in the case of chiral gauge theories where fermions are massless.

Schwinger proper time regularization: This technique involves making use of Laplace transforms and the introducing a regularization parameter to control the divergences of these. It is not currently in popular use.

Lattice field theory regularization: Euclidean spacetime is put onto a finite four-dimensional lattice. It has been used with considerable success to study nonperturbative QCD. It is a first principles approach to nonperturbative studies of quantum field theory in that it is systematically improvable. Lattice gauge theory is a gauge invariant form of cut-off regularization. Gauge invariance is the most important symmetry to maintain in any regularization of gauge theories. The lattice violates rotational and translational invariance since it uses a finite spacetime volume, but these are recovered in the continuum and infinite volume limits. The treatment of chiral symmetry on the lattice requires some care. A introduction to lattice QCD is given in Sec. 9.2.5.

Dimensional regularization

(See Chapter 8, Sec 8.6.2)

In $d < 4$ dimensions logarithmically divergent integrals such as Eq. (8.6.5) are convergent without the Pauli-Villars regulator or any form of cut-off. We define

$$d \equiv 4 - \epsilon \tag{8.6.13}$$

so that $\epsilon \equiv 4-d$. Beware that the choice $2\epsilon = 4-d$ is also common and used in some texts. We have $\Lambda \sim 1/\epsilon$ as the ultraviolet regulator, where $\epsilon \to 0^+$ corresponds to $\Lambda \to \infty$ and is taken at the end of calculations. Because we can make shifts in the momentum integral then Eq. (8.5.20) remains valid and $q_\nu \Pi^{\mu\nu} = 0$ as required.

Similarly in d-dimensions we find for the QED Lagrangian density of Eq. (7.6.16) $[A^\mu] = \mathsf{M}^{(d-2)/2}$, $[\psi] = \mathsf{M}^{(d-1)/2}$, $[m] = \mathsf{M}$ and $[q_c] = \mathsf{M}^{d-(d-1)-(d-2)/2} = \mathsf{M}^{(4-d)/2} = \mathsf{M}^{\epsilon/2}$. Using an arbitrary mass scale μ to keep q_c dimensionless in natural units we replace $q_c \to \mu^{(4-d)/2} q_c = \mu^{\epsilon/2} q_c$. The d-dimensional Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \mu^{(4-d)/2} q_c \bar{\psi} A\!\!\!/ \psi \\ &= \bar{\psi}(iD\!\!\!\!/ - m)\psi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} , \end{aligned} \tag{8.6.15}$$

where in d-dimensions the covariant derivative is $D^\mu = \partial^\mu + i\mu^{(4-d)/2} q_c A^\mu$. In Yukawa theory we similarly have $g \to \mu^{(4-d)/2} g = \mu^{\epsilon/2} g$.

Dimensional regularization

(See Chapter 8, Sec 8.6.2)

In $d < 4$ dimensions logarithmically divergent integrals such as Eq. (8.6.5) are convergent without the Pauli-Villars regulator or any form of cut-off. We define

$$d \equiv 4 - \epsilon \tag{8.6.13}$$

so that $\epsilon \equiv 4-d$. Beware that the choice $2\epsilon = 4-d$ is also common and used in some texts. We have $\Lambda \sim 1/\epsilon$ as the ultraviolet regulator, where $\epsilon \to 0^+$ corresponds to $\Lambda \to \infty$ and is taken at the end of calculations. Because we can make shifts in the momentum integral then Eq. (8.5.20) remains valid and $q_\nu \Pi^{\mu\nu} = 0$ as required.

In natural units the action, $S[\phi] = \int d^d x\, \mathcal{L}$, for any theory must be dimensionless in d dimensions since we exponentiate it in the path integral. Since $[d^d x] = \mathsf{M}^{-d}$ then we require $[\mathcal{L}] = \mathsf{M}^d$. Since $[\partial_\mu] = \mathsf{M}$ then we can deduce the dimension of any field from its kinetic term. For example, for scalars we must have $[\partial_\mu \phi \partial^\mu \phi] = \mathsf{M}^d$ and so $[\phi] = \mathsf{M}^{(d-2)/2}$. For a fermion, since we must have $[\bar{\psi} \partial\!\!\!/ \psi] = \mathsf{M}^d$, then $[\psi] = \mathsf{M}^{(d-1)/2}$. Since particle masses m appear once for each ∂_μ in the kinetic terms of theories, then $[m] = [\partial_\mu] = \mathsf{M}$. In ϕ^4 theory in d-dimensions the interaction term has dimension $[(\lambda/4!)\phi^4] = \mathsf{M}^d$ and so $[\lambda] = \mathsf{M}^{d-2(d-2)} = \mathsf{M}^{4-d}$. However, since we prefer to keep the coupling dimensionless as it is in $d = 4$ we introduce an *arbitrary mass scale*, μ, with $[\mu] = \mathsf{M}$ and make the replacement $\lambda \to \mu^{4-d}\lambda = \mu^\epsilon \lambda$ so that

Dimensional regularization

(See Chapter 8, Sec 8.6.2)

now $[\lambda] = \mathsf{M}^0$. In summary, for ϕ^4 theory in d-dimensions the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu - \tfrac{1}{2}m^2\phi^2 - (\mu^{4-d}\lambda/4!)\phi^4 \,, \tag{8.6.14}$$

where λ is dimensionless. In the limit $\epsilon \to 0^+$ we have $\mu^{4-d} = \mu^\epsilon \to 1$ and so μ and ϵ both become irrelevant in the $\epsilon \to 0^+$ limit for physical quantities in a renormalizable theory. In dimensional regularization the arbitrary mass scale μ can be used as the renormalization scale μ.

Similarly in d-dimensions we find for the QED Lagrangian density of Eq. (7.6.16) $[A^\mu] = \mathsf{M}^{(d-2)/2}$, $[\psi] = \mathsf{M}^{(d-1)/2}$, $[m] = \mathsf{M}$ and $[q_c] = \mathsf{M}^{d-(d-1)-(d-2)/2} = \mathsf{M}^{(4-d)/2} = \mathsf{M}^{\epsilon/2}$. Using an arbitrary mass scale μ to keep q_c dimensionless in natural units we replace $q_c \to \mu^{(4-d)/2}q_c = \mu^{\epsilon/2}q_c$. The d-dimensional Lagrangian density is

$$\begin{aligned}\mathcal{L} &= \bar\psi(i\partial\!\!\!/ - m)\psi - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \mu^{(4-d)/2}q_c\bar\psi A\!\!\!/\psi \\ &= \bar\psi(iD\!\!\!\!/ - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,\end{aligned} \tag{8.6.15}$$

where in d-dimensions the covariant derivative is $D^\mu = \partial^\mu + i\mu^{(4-d)/2}q_cA^\mu$. In Yukawa theory we similarly have $g \to \mu^{(4-d)/2}g = \mu^{\epsilon/2}g$.

Dimensional regularization

(See Chapter 8, Sec 8.7.4)

Interpretation of μ as a physical scale: Renormalized perturbation theory will converge most quickly in powers of the coupling q_c when the loop corrections are as small as possible. Consider the one-loop result for $\Pi(q^2)$ above for $Q^2 = -q^2 \gtrsim m^2$. The effects of the loop correction are smallest when $\mu^2 \sim Q^2$. This pattern continues at higher loops with higher powers of q_c^2. So *we should choose μ to be similar to the characteristic momentum scale relevant to the physical process* so that we have optimal convergence at a given order in the renormalized perturbation theory. In this way we associate μ with the characteristic scale Q^2 in a physical process. So, while the on-shell scheme directly connects with physical mass and charge, when Q^2 is large perturbation theory will converge best in schemes like the MS or $\overline{\text{MS}}$ with the renormalization scale μ chosen such that $\mu^2 \sim Q^2$.

Renormalizing with dimensional regularization: To implement calculations of renormalized perturbation theory with dimensional regularization requires a considerable amount of machinery. Some of these tricks and needed results are shown on the following pages. The calculations require some effort.

Feynman parameterization

(See Appendix, Sec A.5)

To simplify the evaluation of loop integrals in perturbation theory there are a number of useful identities as we will see in Sec. A.6. The simplest of these is

$$\frac{1}{A_1A_2}=\int_0^1 dx_1dx_2\,\frac{\delta(x_1+x_2-1)}{[x_1A_1+x_2A_2]^2}=\int_0^1 dx\,\frac{1}{[xA_1+(1-x)A_2]^2}\,. \tag{A.5.1}$$

The parameters x_1, x_2, and x are all referred to as *Feynman parameters*. The proof of this result is straightforward from right to left using

$$\int_0^1 dx\,\frac{1}{[ax+b]^2}=\frac{1}{a}\int_b^{a+b} dy\,\frac{1}{y^2}=\frac{1}{a}\left[-\frac{1}{y}\right]_b^{a+b}=\frac{1}{(a+b)b}\,, \tag{A.5.2}$$

where we have used the change of variable $y = ax + b$ and where we identify $a = (A_1 - A_2)$ and $b = A_2$. After differentiating Eq. (A.5.1) $n_1 - 1$ times with respect to A_1 and $n_2 - 1$ times with respect to A_2 we immediately obtain

$$\frac{1}{A_1^{n_1}A_2^{n_2}}=\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1 dx_1dx_2\,\delta(x_1+x_2-1)\,\frac{x_1^{n_1-1}x_2^{n_2-1}}{[x_1A_1+x_2A_2]^{n_1+n_2}}\,, \tag{A.5.3}$$

since

$$\frac{d^{n_1+n_2-2}}{dA_1^{n_1-1}dA_2^{n_2-1}}\frac{1}{A_1A_2}=(-1)^{n_1+n_2-2}\,\frac{(n_1-1)!(n_2-1)!}{A_1^{n_1}A_2^{n_2}}\,, \tag{A.5.4}$$

$$\frac{d^{n_1+n_2-2}}{dA_1^{n_1-1}dA_2^{n_2-1}}\frac{1}{[x_1A_1+x_2A_2]^2}=(-1)^{n_1+n_2-2}\,\frac{(n_1+n_2-1)!\,x_1^{n_1-1}x_2^{n_2-1}}{[x_1A_1+x_2A_2]^{n_1+n_2}}\,.$$

Dimensional regularization

(See Appendix, Sec A.6)

Evaluating loop integrals in d-dimensions: The following Minkowski-space results are useful,

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-a^2)^\alpha} = \left[\frac{i(-1)^\alpha}{(4\pi)^{d/2}}\right] \frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)} \left(\frac{1}{a^2}\right)^{\alpha-\frac{d}{2}} \tag{A.6.4}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-a^2)^\alpha} = -\left[\frac{i(-1)^\alpha}{(4\pi)^{d/2}}\right] \frac{d}{2} \frac{\Gamma(\alpha-\frac{d}{2}-1)}{\Gamma(\alpha)} \left(\frac{1}{a^2}\right)^{\alpha-\frac{d}{2}-1} \tag{A.6.5}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu\ell^\nu}{(\ell^2-a^2)^\alpha} = -\left[\frac{i(-1)^\alpha}{(4\pi)^{d/2}}\right] \frac{g^{\mu\nu}}{2} \frac{\Gamma(\alpha-\frac{d}{2}-1)}{\Gamma(\alpha)} \left(\frac{1}{a^2}\right)^{\alpha-\frac{d}{2}-1} \tag{A.6.6}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{(\ell^2)^2}{(\ell^2-a^2)^\alpha} = \left[\frac{i(-1)^\alpha}{(4\pi)^{d/2}}\right] \frac{d(d+2)}{4} \frac{\Gamma(\alpha-\frac{d}{2}-2)}{\Gamma(\alpha)} \left(\frac{1}{a^2}\right)^{\alpha-\frac{d}{2}-2} \tag{A.6.7}$$

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu\ell^\nu\ell^\rho\ell^\sigma}{(\ell^2-a^2)^\alpha} = \left[\frac{i(-1)^\alpha}{(4\pi)^{d/2}}\right] \frac{1}{4} \frac{\Gamma(\alpha-\frac{d}{2}-2)}{\Gamma(\alpha)} \left(\frac{1}{a^2}\right)^{\alpha-\frac{d}{2}-2} \times \left[g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right]. \tag{A.6.8}$$

These results are most easily shown by Wick rotating to Euclidean space, where $d^d\ell \to i d^d\ell^E$, $(\ell^2-a^2) \to -(\ell^{E2}+a^2)$ as explained in Sec. A.4. Useful formulae for arriving at the above results for convergent integrals are

$$\int d^d\ell\, \ell^\mu\ell^\nu f(\ell^2) = \int d^d\ell\, \tfrac{1}{d}\, \ell^2 g^{\mu\nu} f(\ell^2), \tag{A.6.9}$$

$$\int d^d\ell\, \ell^\mu\ell^\nu\ell^\rho\ell^\sigma f(\ell^2) = \int d^4\ell\, \tfrac{1}{(d+2)d} \left(\ell^2\right)^2 \left[g^{\mu\nu}g^{\rho\sigma}+g^{\mu\nu}g^{\rho\sigma}+g^{\mu\nu}g^{\rho\sigma}\right] f(\ell^2). \tag{A.6.10}$$

Note that for an odd number of momenta $\ell^\mu\ell^\nu\cdots$ such integrals vanish, while for any even number, $2n$, we must have a completely symmetrized sum of n factors of the metric tensor $g^{\mu\nu}$, where the appropriate normalization can be deduced.

Dirac algebra in $d = 4 - \epsilon$ dimensions

(See Appendix, Sec A.6)

When extending from integer, $n \in \mathbb{Z}$, to noninteger spacetime dimensions, $d \in \mathbb{R}$, we define the contraction identities with n replaced by $d = 4 - \epsilon$,

$$\gamma^\mu \gamma_\mu = (4-\epsilon) I\,, \quad \gamma^\mu \not{k} \gamma_\mu = -(2-\epsilon)\not{k}\,, \quad \gamma^\mu \not{k}\not{\ell}\gamma_\mu = 4k\cdot\ell - \epsilon \not{k}\not{\ell}\,,$$
$$\gamma^\mu \not{k}\not{\ell}\not{p}\gamma_\mu = -2\not{p}\not{\ell}\not{k} + \epsilon\not{k}\not{\ell}\not{p}\,. \tag{A.6.19}$$

The trace identities can be generalized to $\mathrm{tr} I = f(d)$ and $\mathrm{tr}(\gamma^\mu\gamma^\nu) = f(d) g^{\mu\nu}$, where $f(d)$ is some arbitrary smooth function of d (Itzykson and Zuber, 1980) such that $f(n) = n_{\mathrm{matrix}}$. We will almost exclusively be interested in analytically continuing from $n = 4$ dimensions to $d = 4 - \epsilon$ dimensions. Without loss of generality it is convenient to choose $f(d) = 4$ for $d = 4 - \epsilon$ so that, e.g.,

$$\mathrm{tr} I = 4\,, \quad \mathrm{tr}\gamma^\mu = 0 \quad \text{and} \quad \mathrm{tr}(\gamma^\mu\gamma^\nu) = 4 g^{\mu\nu}\,. \tag{A.6.20}$$

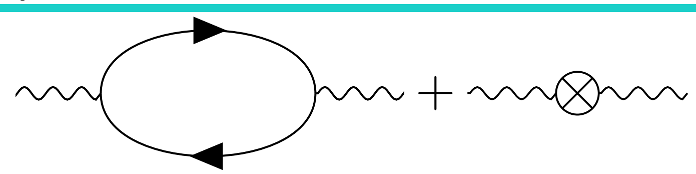
It is important to use $g^{\mu\nu} g_{\nu\mu} = \delta^\mu{}_\mu = d$ before taking $\epsilon \to 0$. These choices have the advantage that *the trace identities in $d = 4 - \epsilon$ dimensions are the same as they are in four-dimensions,* i.e., the identities in Eqs. (A.3.24) to (A.3.29) still apply.

The definition of γ^5 is intrinsically four-dimensional due to its definition in terms of the four-dimensional antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ in Eq. (A.3.5). A variety of recipes for defining γ^5 in d-dimensions exist, (see, e.g., Chanowitz et al. (1979); Fujii et al. (1981) and references therein). However, common practice Muta (1987); Yndúráin (1983) is to simply assume that in d-dimensions there exists some hermitian matrix γ^5 which satisfies

$$\{\gamma^\mu, \gamma^5\} = 0\,, \qquad \gamma^5 = \gamma^{5\dagger}\,, \qquad (\gamma^5)^2 = I\,. \tag{A.6.21}$$

Example calculation of one-loop photon contribution

(See Chapter 8, Sec 8.7.3)

$i\Pi_2^{\mu\nu}(q) =$  (8.7.26)

$$= (-1)(-i\nu^{\epsilon/2}q_c)^2\int\frac{d^d\ell}{(2\pi)^d}\,\mathrm{tr}\left[\frac{i(\not\ell+m)\gamma^\mu i(\not\ell+\not q+m)\gamma^\nu}{[(\ell+q)^2-m^2+i\epsilon][\ell^2-m^2+i\epsilon]}\right]-i(q^2g^{\mu\nu}-q^\mu q^\nu)\delta_3$$

$$= -4\nu^\epsilon q_c^2\int\frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^\mu(\ell+q)^\nu+\ell^\nu(\ell+q)^\mu+g^{\mu\nu}[m^2-\ell\cdot(\ell+q)]}{[(\ell+q)^2-m^2+i\epsilon][\ell^2-m^2+i\epsilon]}-i(q^2g^{\mu\nu}-q^\mu q^\nu)\delta_3$$

$$= -4\nu^\epsilon q_c^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\,\frac{\ell^\mu(\ell+q)^\nu+\ell^\nu(\ell+q)^\mu+g^{\mu\nu}[m^2-\ell\cdot(\ell+q)]}{\{\ell^2+2xq\cdot\ell+xq^2-m^2+i\epsilon\}^2}-i(q^2g^{\mu\nu}-q^\mu q^\nu)\delta_3$$

$$= -4\nu^\epsilon q_c^2\int_0^1 dx\int\frac{d^d\ell'}{(2\pi)^d}\,\frac{2\ell'^\mu\ell'^\nu-g^{\mu\nu}\ell'^2-2x(1-x)q^\mu q^\nu+g^{\mu\nu}[m^2+x(1-x)q^2]+\mathcal{O}(\ell')}{\{\ell'^2+x(1-x)q^2-m^2+i\epsilon\}^2}$$
$$-i(q^2g^{\mu\nu}-q^\mu q^\nu)\delta_3$$

$$= \frac{4i\nu^\epsilon q_c^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})[a^2g^{\mu\nu}+2x(1-x)q^\mu q^\nu-g^{\mu\nu}(2m^2-a^2)\}]}{\{a^2\}^{2-(d/2)}}-i(q^2g^{\mu\nu}-q^\mu q^\nu)\delta_3$$

$$= \frac{4i\nu^\epsilon q_c^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})2x(1-x)[-q^2g^{\mu\nu}+q^\mu q^\nu]}{\{m^2-x(1-x)q^2\}^{2-(d/2)}}-i(q^2g^{\mu\nu}-q^\mu q^\nu)\delta_3$$

$$= -i[q^2g^{\mu\nu}-q^\mu q^\nu]\left[\frac{8q_c^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})\nu^{2-(d/2)}x(1-x)}{\{m^2-x(1-x)q^2\}^{2-(d/2)}}+\delta_3\right]$$

$$= -i[q^2g^{\mu\nu}-q^\mu q^\nu]\left[\frac{2}{\pi}\frac{q_c^2}{4\pi}\int_0^1 dx\,x(1-x)\left\{\frac{2}{\epsilon}-\ln\left(\frac{m^2-x(1-x)q^2}{\tilde\nu^2}\right)\right\}+\delta_3\right]$$

$$\equiv -i[q^2g^{\mu\nu}-q^\mu q^\nu]\Pi_2(q^2)\,,$$

Note: Dimensional regularization has preserved the Ward identity $q_\mu\Pi^{\mu\nu}=0$ as we wanted!

Running coupling and running mass in QED

(See Chapter 8, Sec 8.7.5)

FInally: We state without the detailed proofs two very important results from one-loop calculations in QED. First the running coupling $\alpha(\mu)$ increasing logarithmically with scale μ below and on the following page running mass $m(\mu)$ logarithmically decreasing with scale μ.

$$\begin{aligned}\alpha(\mu_b) &= \frac{1}{[\beta_0/4\pi]\{[4\pi/\beta_0\alpha(\mu_a)] - \ln(\mu_a^2) + \ln(\mu_b^2)\}} \\ &= \frac{4\pi}{\beta_0\{\ln(\mu_b^2) - \ln(\Lambda_{\rm QED}^2)\}} = \frac{4\pi}{\beta_0 \ln(\mu_b^2/\Lambda_{\rm QED}^2)} = \frac{2\pi}{\beta_0 \ln(\mu_b/\Lambda_{\rm QED})}, \qquad (8.7.72)\end{aligned}$$

where the first line shows that $[4\pi/\beta_0\alpha(\mu_a)] - \ln(\mu_a^2)$ must be independent of μ_a and so it can be replaced with $[4\pi/\beta_0\alpha(m_e)] - \ln(m_e^2) = -\ln(\Lambda_{\rm QED})$. This is a very powerful one-loop result that shows that: (i) since $\beta_0 = -4/3 < 0$ and $\mu < \Lambda_{QED}$ then $\alpha(\mu)$ increases with increasing renormalization scale; and (ii) the coupling at scale μ is determined by the location of the Landau pole, $\Lambda_{\rm QED}$. Since $\alpha(\mu) = e(\mu)^2/4\pi \geq 0$ and $\beta_0 < 0$ then Eq. (8.7.72) can only be valid when $\mu < \Lambda_{\rm QED}$, which means that $\ln(\mu/\Lambda_{\rm QED}) < 0$ for all relevant μ. The effective replacement of the dimensionless coupling α with a dimensionful scale $\Lambda_{\rm QED}$ is sometimes referred to as *dimensional transmutation*. Using this formula we find that the value of the fine structure constant at the Z-boson mass is approximately $\alpha_{\rm naive}(M_Z) \simeq 1/134$. However, at high-momentum scales we should also include the photon coupling to more massive charged particles including the μ and τ leptons, the quarks and the $W^\pm$ bosons. The measured value at M_Z is $\alpha(M_Z) \simeq 1/128$.

Note: $\Lambda_{\rm QED} \sim 10^{286}$ eV and so we are not going to easily access this scale!

Running coupling and running mass in QED

(See Chapter 8, Sec 8.7.5)

$$\frac{m(\mu_b)}{m(\mu_a)} = \left[\frac{\ln(\mu_b^2/\Lambda_{\rm QED}^2)}{\ln(\mu_a^2/\Lambda_{\rm QED}^2)}\right]^{-3/\beta_0} = \left[\frac{\alpha(\mu_a)}{\alpha(\mu_b)}\right]^{-3/\beta_0} = \left[\frac{\alpha(\mu_a)}{\alpha(\mu_b)}\right]^{9/4}, \qquad (8.7.75)$$