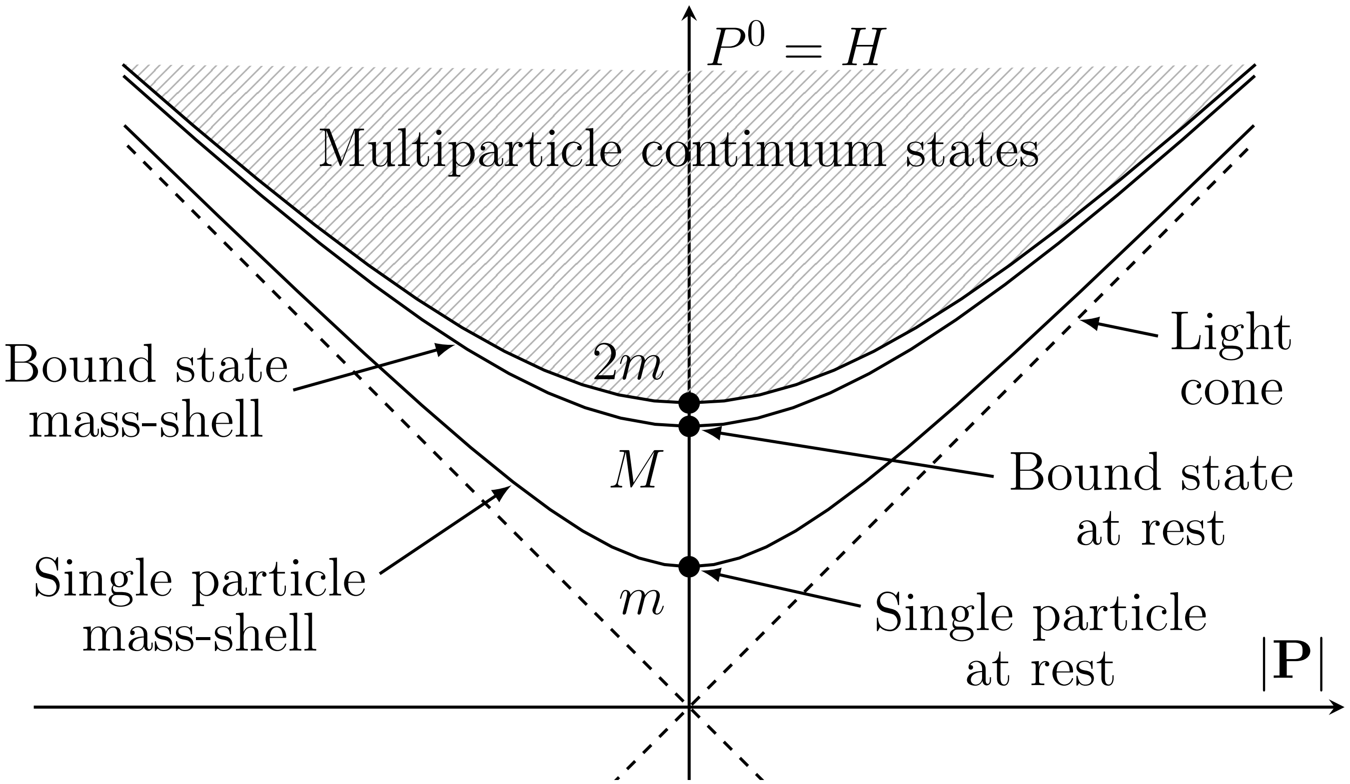
Lecture 5: Interacting field theories, Feynman diagrams, cross sections,QED tree-level calculations

Physical spectrum of states

(See Chapter 7, Sec 7.1)



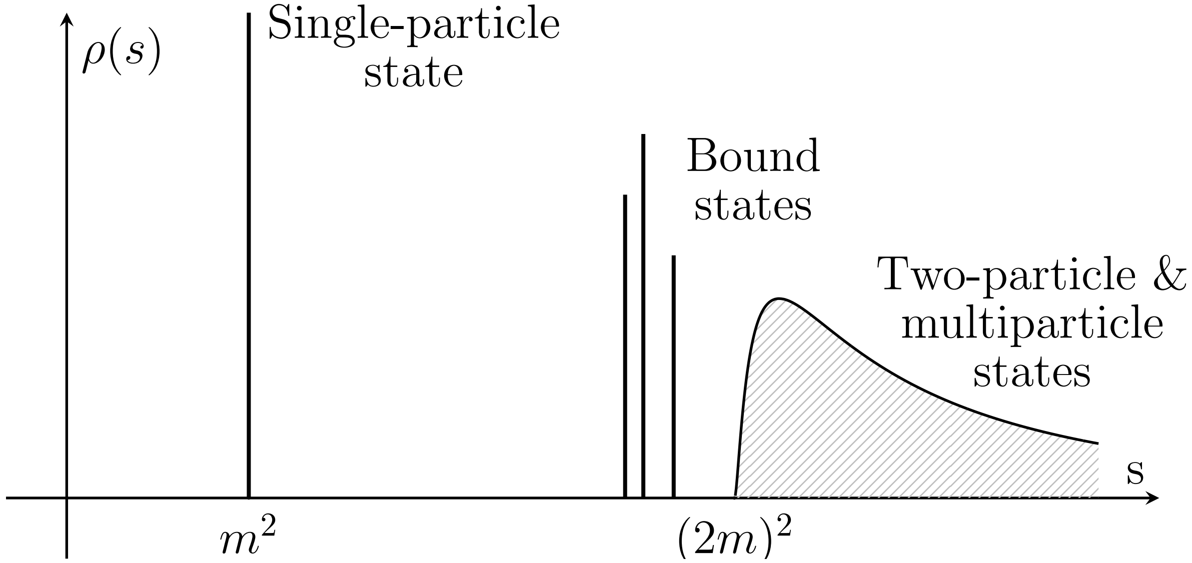
Physical spectrum of states: We can build a mixed Fock space in terms of the full vacuum $|\Omega\rangle$, physical singe-particle states and the different bound states of the full interacting theory. This is a Fock space constructed from free single particles of mass m and free bound states of masses M_1, M_2 and so on. Denote this Fock space as V_{Fock} and the full Hilbert space of the interacting theory as V_{full}. Let $|\mathbf{p}\rangle$ denote a single particle or a stable bound state in an on-shell momentum eigenstate in the full theory. Then clearly for all $|\mathbf{p}\rangle$ we have

$$|\Omega\rangle, |\mathbf{p}\rangle \in V_{\text{Fock}} \quad \text{and} \quad |\Omega\rangle, |\mathbf{p}\rangle \in V_{\text{full}}\,.$$

Cluster decomposition principle: In the above discussion we are implicitly assuming that when stable particles and/or stable bound states have spacelike separations approaching infinity, then we can neglect their influence on each other. This is a familiar property of physical systems. Were it not the case, then to describe the behavior of any system we would have to describe the behavior of the whole universe. All systems of physical relevance satisfy this principle.

Källen-Lehmann spectral representation

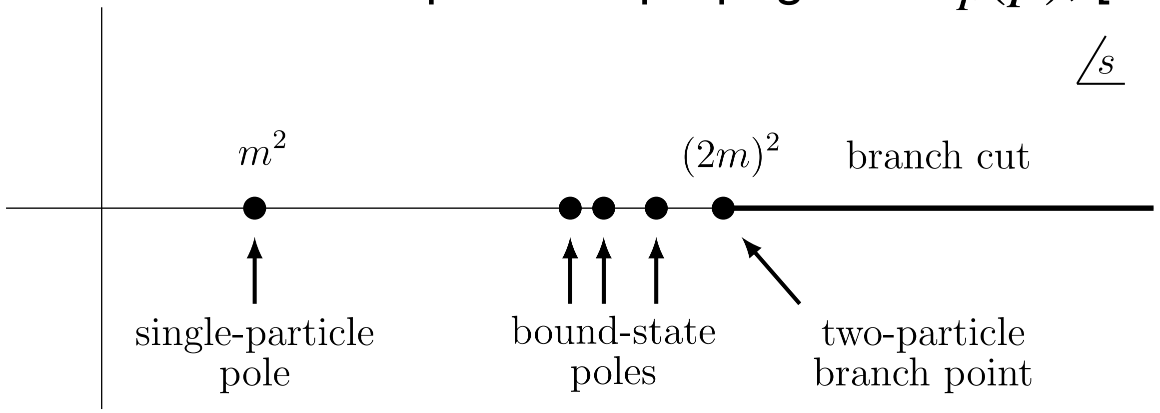
(See Chapter 7, Secs 7.1 and 7.2)



Källen-Lehmann spectral representation: We can define the full propagator of the interacting theory as

$$D_F(x-y) \equiv \langle \Omega | T\hat{\phi}(x)\hat{\phi}(y) | \Omega \rangle = \int \frac{d^4p}{(2\pi)^4} \left[\int_0^\infty ds \, \frac{i\rho(s)}{p^2 - s + i\epsilon} \right] e^{-ip\cdot(x-y)} \equiv \int \frac{d^4p}{(2\pi)^4} D_F(p) e^{-ip\cdot(x-y)},$$

where the last line defines the momentum-space full propagator $D_F(p)$, [see the proof of Eq. (7.2.9)]



In the complex s-plane we have poles at single particle and bound state masses and branch cuts starting at multi particle production thresholds. We then have

$$\rho(s) = Z\delta(s-m^2) + [\text{bound state } \delta\text{-functions}]$$
$$+[\text{2-particle part for } s \geq (2m)^2] + [\text{3-particle part for } s \geq (3m)^2] + \cdots \, .$$

Scattering matrix

(See Chapter 7, Sec 7.3)

Definition of the S-matrix: The *Scattering Matrix (S-matrix)* is defined through its matrix elements S_{fi} as

$$S_{fi} \equiv \langle f | \lim_{T\to\infty} \hat{U}(T, -T) | i \rangle = \langle f | \hat{S} | i \rangle ,$$

where to describe scattering $|i\rangle$ and $|f\rangle$ are Heisenberg picture states. In particular they are chosen to be states that describe sets of particles in wave packet states separated by arbitrarily large spatial distances. So if the cluster decomposition principle holds then they approach wave packet states in the non-interacting Fock space, because interactions between the wave packets become negligible at large spatial separations. Then such $|i\rangle$ and $|f\rangle$ in the full theory are asymptotically close to their Fock space versions.

Taking the infinite separation of the wave packets limit *first*, we can then *after that* let the wave packets become sufficiently broad that they approach plane wave states. In summary, we work in the limit where

$$|i\rangle \to |\mathbf{p}_1 \cdots \mathbf{p}_m\rangle^F \text{ and } |f\rangle \to |\mathbf{k}_1 \cdots \mathbf{k}_n\rangle^F$$

with an m-particle initial state of approximate plane waves and an n-particle final state of plane waves both in the Fock space. *The order of the limits is important to remember.*

The Transition Matrix (T-matrix): It is conventional to separate the S-matrix operator $\hat{S}$ into a part where no transition takes place, $\hat{I}$, and a part where transitions occur, $i\hat{T}$,

$$\hat{S} \equiv \hat{I} + i\hat{T} \quad \text{and} \quad S_{fi} = \delta_{fi} + iT_{fi} .$$

The factor of i in the $\hat{T}$ operator definition is conventional. The T_{fi} are the elements of the T-matrix, which is also referred to as the *transition matrix* or *transfer matrix.* We understand that $\delta_{fi} \equiv \langle f | i \rangle$ and assumes unit normalization for our states.

Invariant amplitude

(See Chapter 7, Sec 7.3)

Definition of the invariant amplitude $\mathscr{M}$: Now consider an initial state consisting of two particles with masses m_A and m_B and three-momenta $\mathbf{p}_A$ and $\mathbf{p}_B$ so that $|i\rangle = |\mathbf{p}_A\mathbf{p}_B\rangle$ and a final state consisting of n particles with masses $m_1, \ldots, m_n$ and three-momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$ so that $|f\rangle = |\mathbf{p}_1 \cdots \mathbf{p}_n\rangle$. In a Poincaré-invariant system we have four-momentum conservation and we can factor out a momentum conserving delta-function such that

$$T_{fi} = \langle f|\hat{T}|i\rangle \equiv (2\pi)^4\delta^4(p_A+p_B-\textstyle\sum_{i=1}^n p_i)\mathscr{M}_{fi}\,,$$

where $i=(\mathbf{p}_A,\mathbf{p}_B)$ and $f=(\mathbf{p}_1,\ldots,\mathbf{p}_n)$ and $\mathscr{M}_{fi}$ is called the *invariant amplitude* or *invariant matrix element.* We are initially consider scalar (no spin) particles for simplicity.

Using a simple shorthand we see that in all cases where $f \neq i$ we have

$$|\langle f|\hat{S}|i\rangle|^2 = |\langle f|\hat{T}|i\rangle|^2 = (2\pi)^8\delta^4(0)\delta^4(p_A+p_B-\textstyle\sum_{j=1}^n p_j)|\mathscr{M}_{fi}|^2$$

$$= (V\Delta T)(2\pi)^4\delta^4(p_A+p_B-\textstyle\sum_{j=1}^n p_j)|\mathscr{M}_{fi}|^2\,,$$

where we have used the result that in any regularization of the spacetime volume we have the correspondence $(2\pi)^4\delta^4(0) \to (V\Delta T)$, where V is the finite spatial volume and ΔT is a finite time interval. We emulate the scattering problem in a finite spacetime volume and then take the infinite volume limit, but we will omit that detail here.

The invariant amplitude $\mathscr{M}_{fi}$ plays an essential role in the calculation of physical cross-sections as we will soon see.

Cross-section

(See Chapter 7, Sec 7.3.1)

Classical cross-section: Classically the cross-section A_X of an object is the two-dimensional area it presents in a direction orthogonal to an incident beam of light or particles as shown below.

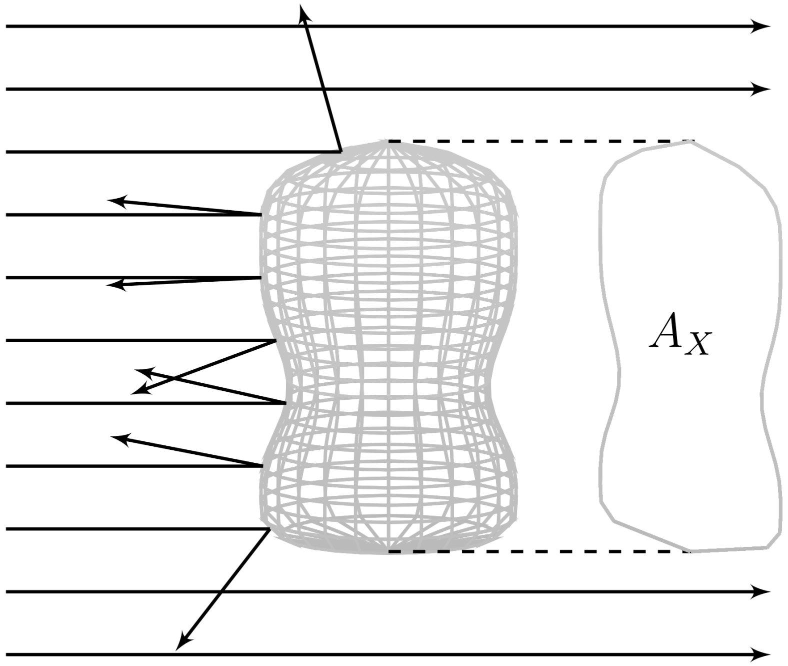
Measuring the cross section: Let Φ denote the flux of the incident beam, which is defined as the number density of the beam ρ multiplied by the beam velocity v,

$$\Phi \equiv \rho v = \text{number of particles/unit time/unit area}$$

Φ is the number of beam particles passing though a unit area orthogonal to the beam per second. The current of incident particles on an area A orthogonal to the beam is $I_{\text{inc}} = \Phi A = \rho v A$.

The number of particles scattered is N_{sc} and the number scattered per unit time by the target is the scattered current, I_{sc}, which is also called the *scattering rate,* R, so by definition

$$R \equiv \frac{dN_{\text{sc}}}{dt} \equiv I_{\text{sc}} \,.$$



The *total cross-section, σ,* is defined as,

$$\sigma \equiv \frac{\#\text{ of scattered particles/sec}}{\#\text{ of incident particles/sec/unit area}} = \frac{I_{\text{sc}}}{\Phi} \,.$$

In classical physics every beam particle that hits the object will scatter.
So we have $I_{\text{sc}} = \Phi A_X$ and so classically we have

$$\sigma = \frac{I_{\text{sc}}}{\Phi} = \frac{\Phi A_X}{\Phi} = A_X \,.$$

which is why we refer to σ as a cross-section *even in quantum mechanics*. It has the units of area.
The total cross-section σ is the effective area of the beam that is scattered. It is also the effective scattering area of the target seen by each particle in the beam.

Differential cross-section

(See Chapter 7, Sec 7.3.1 and 7.3.2)

Diverging cross-sections: Consider a beam of charged particles scattering off of some charged target. Since the Coulomb potential has an infinite range, classically every particle in the beam will be deflected. Beam particles increasingly far from the target will be deflected by decreasing amounts. The "forward scattering" component of the cross-section of the beam then diverges as $\theta \to 0$. Even for finite range interactions the unscattered beam is in the forward direction and so it is not meaningful to measure very forward (or very backward) scattering for this reason. So we typically study the *differential cross-section,* where *final-state particles are **not** moving forward or backward along the beam axis* but instead scatter into some finite solid angle Ω not containing the beam axis.

Differential cross section: We then define the differential cross-section as

$$\frac{d\sigma}{d\Omega} \equiv \frac{\#\text{ of scattered particles/sec/unit solid angle}}{\#\text{ of incident particles/sec/unit area}} = \frac{1}{\Phi}\frac{dI_{\text{sc}}(\theta,\phi)}{d\Omega} .$$

We sometimes also refer to $d\sigma$ as the differential of the cross-section. We obviously have

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int_0^\pi d\theta \int_0^{2\pi} \sin\theta d\phi \frac{d\sigma}{d\Omega}(\theta,\phi) .$$

Similarly the variation of the cross-section with any combination of kinematic variables is said to be a differential cross-section, e.g., $d\sigma/d\theta, d\sigma/dE, d^2\sigma/dxdy$ for any kinematic variables x and y and so on.

Relating the cross-section to the S-matrix and the invariant amplitude $\mathscr{M}$: It is not possible to do this justice here. A thorough discussion is given in Sec. 7.3.2. We simply give the result, which is that

$$\sigma \equiv \sigma_{AB} = \sum_{n=1,2,\dots} \left[\int \prod_{k=1}^{n} \frac{d^3 p_k}{(2\pi)^3 2E_{\mathbf{p}_j}} \right] \frac{(2\pi)^4 \delta^4(p_A + p_B - \sum_{j=1}^n p_j) |\mathscr{M}_{fi}|^2}{|\mathbf{v}_A - \mathbf{v}_B| 4E_{\mathbf{p}_A} E_{\mathbf{p}_B}} ,$$ includes *all* final states f

where we are assuming that a beam of A particles with velocity $\mathbf{v}_A$ collides collinearly with a beam of B particles with velocity $\mathbf{v}_B$, i.e., $|\mathbf{v}_A - \mathbf{v}_B|$ is the relative speed of the oppositely approaching beams.

Two-body scattering & Mandelstam variables

(See Chapter 7, Sec 7.3.4)

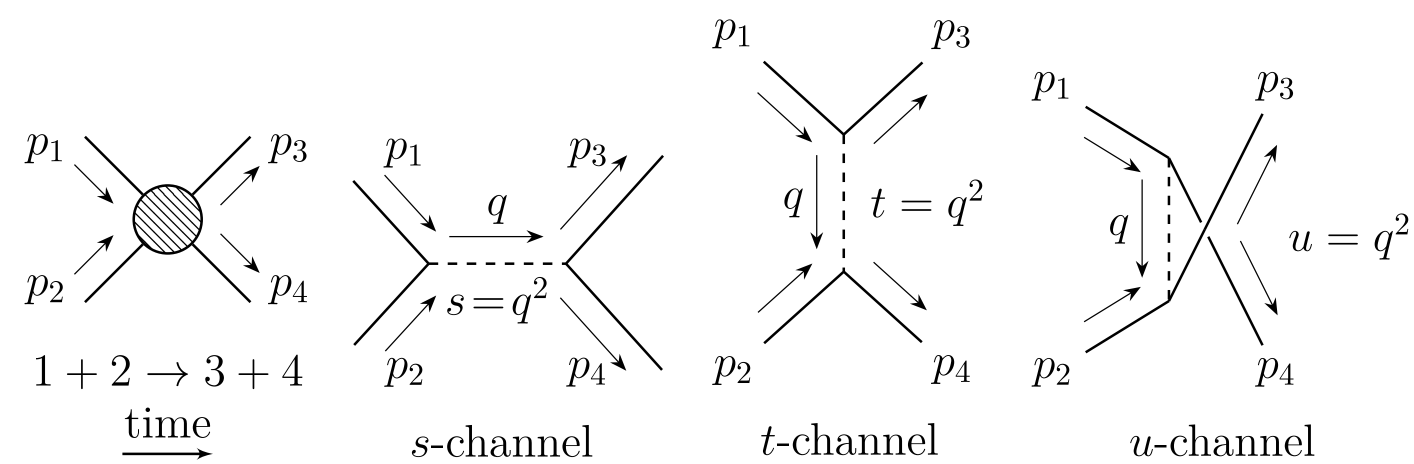


Figure 7.10

Two-body scattering and Mandelstam variables: $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$ and $u=(p_1-p_4)^2$.

LIPS: *Lorentz invariant phase space (LIPS)* consists of products of the Lorentz invariant combinations $d^3p/E_{\mathbf{p}}$ and the Lorentz invariant four-momentum delta function $\delta^4(\cdots)$ and its companion factor $(2\pi)^4$.

we have for the s-channel $q^\mu = (p_1^\mu + p_2^\mu) = (p_3^\mu + p_4^\mu)$ and $s = q^2$, for the t-channel we have $q^\mu = (p_1^\mu - p_3^\mu) = (p_4^\mu - p_2^\mu)$ and $t = q^2$, and for the u-channel we have $q^\mu = (p_1^\mu - p_4^\mu) = (p_3^\mu - p_2^\mu)$ and $u = q^2$. Note that $s = p_1^2 + p_s^2 + 2p_1 \cdot p_2$, $t = p_1^2 + p_3^2 - 2p_1 \cdot p_3$ and $u = p_1^2 + p_4^2 - 2p_1 \cdot p_4$. Summing these we find

$$
\begin{aligned}
s+t+u &= m_1^2+m_2^2+m_3^2+m_4^2+2p_1^2+2p_1\cdot p_2-2p_1\cdot p_3-2p_1\cdot p_4 \\
&= m_1^2+m_2^2+m_3^2+m_4^2+2p_1\cdot(p_1+p_2-p_3-p_4) \\
&= m_1^2+m_2^2+m_3^2+m_4^2\,. \qquad (7.3.76)
\end{aligned}
$$

For two particles scattering into a two-particle final state, $1+2\to 3+4$, we have in any collinear inertial frame,

$$
d\sigma = \frac{|\mathcal{M}_{fi}|^2}{4E_1E_2|\boldsymbol{v}_1-\boldsymbol{v}_2|}d\Pi_2^{\rm LIPS} = \frac{|\mathcal{M}_{fi}|^2}{4F}d\Pi_2^{\rm LIPS}\,,
$$

$$
d\Pi_2^{\rm LIPS} \equiv (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)[d^3p_3/(2\pi)^3 2E_{\mathbf{p}_3}][d^3p_4/(2\pi)^3 2E_{\mathbf{p}_4}]\,. \qquad (7.3.77)
$$

Two-body scattering

(See Chapter 7, Sec 7.3.4)

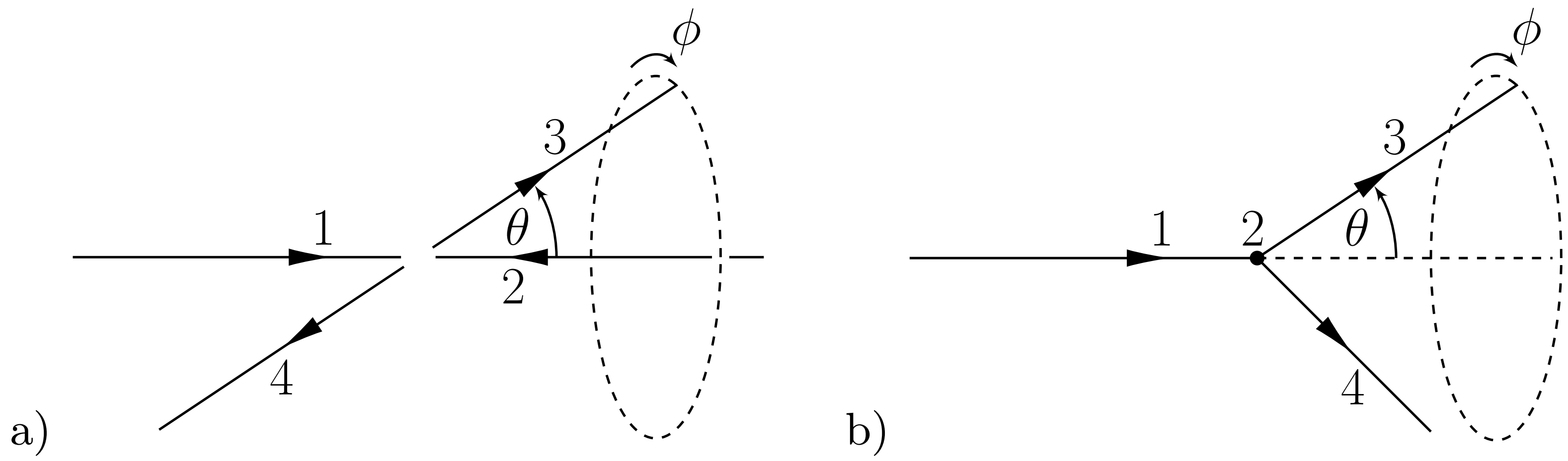


Figure 7.11 Two-body scattering: a) in the Center of Momentum (CM) frame; and b) in the fixed-target frame.

Center of Momentum (CM) frame: In this frame we have by definition

$p_i^{\mathrm{cm}} \equiv |\mathbf{p}_1| = |\mathbf{p}_2|$ and $p_f^{\mathrm{cm}} \equiv |\mathbf{p}_3| = |\mathbf{p}_4|$

and it follows that

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = (E_1 + E_2)^2 = E_{\mathrm{cm}}^2 .$$

Defining $d\Omega = d\theta \sin\theta \, d\phi = d\phi \, d(\cos\theta)$ we obtain the form

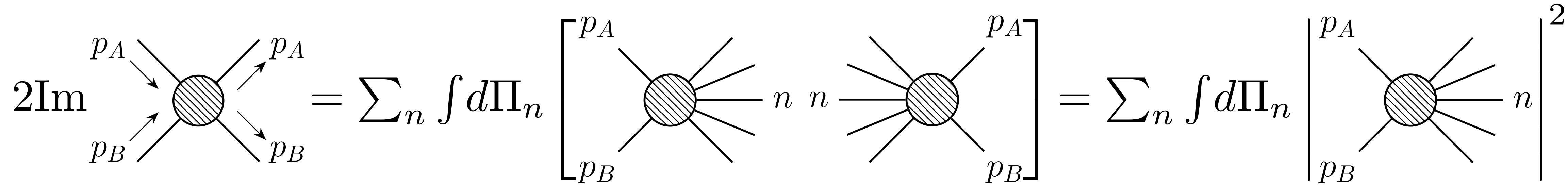
$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{cm}} = \frac{|\mathscr{M}_{fi}|^2}{4p_i^{\mathrm{cm}}\sqrt{s}} \frac{p_f^{\mathrm{cm}}}{16\pi^2\sqrt{s}} = \frac{1}{64\pi^2 s} \frac{p_f^{\mathrm{cm}}}{p_i^{\mathrm{cm}}} |\mathscr{M}_{fi}|^2 .$$

Fixed-target (laboratory) frame: In the fixed target frame, where particle 1 is the beam particle and particle 2 is the target particle at rest we have $\mathbf{p}_2 = 0$. In that case we find

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{p_1 m_2} \frac{p_3^2}{p_3(m_2 + E_{\mathbf{p}_1}) - p_1 E_{\mathbf{p}_3} \cos\theta} |\mathscr{M}_{fi}|^2$$ where here $p_1 \equiv |\mathbf{p_1}|$ and $p_3 \equiv |\mathbf{p_3}|$.

Unitarity of the S-matrix & the Optical theorem

(See Chapter 7, Sec 7.2)



Unitarity of the S-matrix: Consider any complete orthonormal basis of the full Hilbert space with basis vectors $|\,i\rangle$, then $\langle j\,|\,k\rangle = \delta_{jk}$ and $\sum_j |\,j\rangle\langle j\,| = I$. Then $S^\dagger = S^{-1}$ and so S is unitary since

$$(S^\dagger S)_{jk} = \sum_\ell S^\dagger_{j\ell} S_{\ell k} = \lim_{T\to\infty} \sum_\ell \langle j\,|\,\hat{U}(T,-T)^\dagger\,|\,\ell\rangle\langle \ell\,|\,\hat{U}(T,-T)\,|\,k\rangle$$

$$= \lim_{T\to\infty} \langle j\,|\,\hat{U}(T,-T)^\dagger \hat{U}(T,-T)\,|\,k\rangle = \langle j\,|\,k\rangle = \delta_{jk}\,.$$

Since $I = S^\dagger S = (I - iT^\dagger)(I - iT) = I + i(T - T^\dagger) + T^\dagger T$ then

$$-i(T - T^\dagger) = T^\dagger T \qquad \Rightarrow \qquad -i(T_{jk} - T^*_{kj}) = \sum_\ell T^*_{\ell j} T_{\ell k}\,.$$

Then choosing $j = k$ we find the important result that $2\,\mathrm{Im}\,T_{jj} = \sum_\ell |\,T_{\ell j}|^2$. Note that we sum over the full basis but we can choose any specific $|\,j\rangle$ or linear combination of interest.

Moving to our continuous basis the orthonormality and completeness become

$$\langle j\,|\,k\rangle = \delta_{jk} \quad \to \quad \langle \mathbf{p}_1\cdots\mathbf{p}_n\,|\,\mathbf{q}_1\cdots\mathbf{q}_m\rangle = \delta_{nm}(2\pi)^{3n}(2E_{\mathbf{p}_1})\cdots(2E_{\mathbf{p}_n})\delta^3(\mathbf{p}_1 - \mathbf{q}_1)\cdots\delta^3(\mathbf{p}_n - \mathbf{q}_n)\,,$$

$$\hat{I} = \sum_\ell |\,\ell\rangle\langle \ell\,| \quad \to \quad \hat{I} = \sum_n \int \Big[\prod_{k=1}^n \frac{d^3p_k}{(2\pi)^3 2E_{\mathbf{p}_k}}\Big]\,|\,\mathbf{p}_1\cdots\mathbf{p}_n\rangle\langle \mathbf{p}_1\cdots\mathbf{p}_n\,|\,.$$

We then arrive at the result illustrated in the above figure. It can be used to simplify some calculations.

$$2\,\mathrm{Im}\,\mathscr{M}_{p_Ap_B\to p_Ap_B} = \sum_n \int d\Pi_n^{\mathrm{LIPS}}\,|\,\mathscr{M}_{p_Ap_B\to p_1\cdots p_n}|^2\,. \qquad \leftarrow \qquad \textbf{the Optical Theorem}$$

Invariant amplitudes from Feynman diagrams

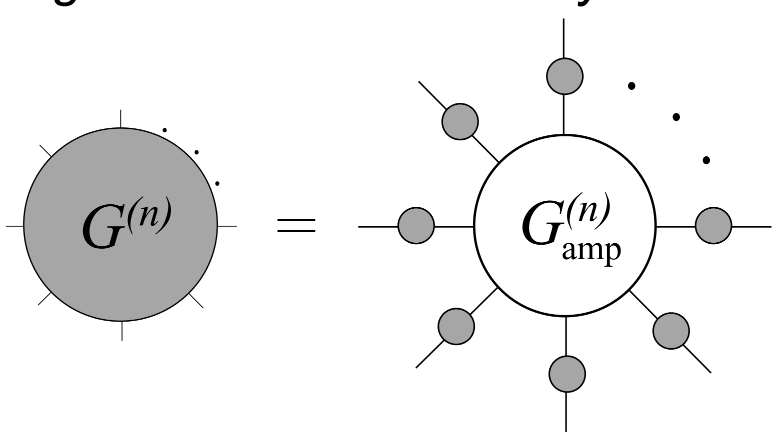
(See Chapter 7, Secs 7.4 and 7.5)

Calculating Green's functions with Feynman diagrams: See Sec. 7.4 for full details. With some effort it is possible to show that

$$\langle\Omega|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|\Omega\rangle = \frac{\int\mathscr{D}\phi\,\phi(x_1)\cdots\phi(x_n)\,e^{iS[\phi]}}{\int\mathscr{D}\phi\,e^{iS[\phi]}} = \frac{(i)^n\delta^k}{\delta j(x_1)\cdots\delta j(x_n)}Z[j]\bigg|_{j=0}$$

$$= \sum_\alpha C_\alpha = \left[\begin{array}{c}\text{sum of all connected Feynman}\\ \text{diagrams with n external points}\end{array}\right] \equiv G^{(n)}(x_1,\cdots,x_n)\,.$$

Calculating invariant amplitudes with Feynman diagrams: Then using the *Lehmann-Symanzik-Zimmermann (LSZ) formalism* it follows that *we obtain invariant amplitudes $\mathscr{M}_{fi}$ by amputating the external legs of the sum of all Feynman diagrams contributing to the Green's function* for this process.



$G^{(n)}(p_1,\cdots,p_n)$ is the FT of $G^{(n)}(x_1,\cdots,x_n)$

$$G^{(n)}_{\mathrm{amp}}(p_1,\cdots,p_n) = D_F^{-1}(p_1)\cdots D_F^{-1}(p_n)\,G^{(n)}(p_1,\cdots,p_n)$$

We then use LSZ to convert the $G^{(n)}_{\mathrm{amp}}$ into $\mathscr{M}_{fi}$ by attaching appropriate external states for the f and i .

We can not possibly do justice to a proof of the LSZ formalism here but a full derivation and discussion for scalars, fermions and photons is given in Sec. 7.5. Given that it now remains to show how to calculate with Feynman diagrams using the *Feynman rules* that can be derived for each theory of interest. We do not derive the Feynman rules for each theory here, but is is relatively straightforward to do from the interacting theory Lagrangian density $\mathscr{L}$ using the functional integral expressions.

Examples of interacting theories:

(See Chapter 7, Secs 7.4 and 7.5)

Scalar field with quartic self-interaction:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}^{\phi 4} = \mathscr{L}_{\text{KG}} + \mathscr{L}_{\text{int}}^{\phi 4} = \frac{1}{2}\left[\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right] - (\lambda/4!)\phi^4 ,$$

$$H = H_0 + H_{\text{int}}^{\phi 4} = H_{\text{KG}} + H_{\text{int}}^{\phi 4} = H_{\text{KG}} + \int d^3x (\lambda/4!)\phi^4 .$$

Yukawa interaction:

$$\mathscr{L}_{\text{int}}^{\text{Yuk}}(\phi, \bar{\psi}, \psi) = -\mathscr{H}_{\text{int}}^{\text{Yuk}}(\phi, \bar{\psi}, \psi) \equiv -g\bar{\psi}\phi\psi \;\text{ or }\; -g\bar{\psi}i\gamma_5\phi\psi ,$$

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}^{\text{Yuk}} = \mathscr{L}_{\text{KG}} + \mathscr{L}_{\text{Dirac}} + \mathscr{L}_{\text{int}}^{\text{Yuk}} = \mathscr{L}_{\text{KG}} + \bar{\psi}(i\partial\!\!\!/ - m_f)\psi + \mathscr{L}_{\text{int}}^{\text{Yuk}} ,$$

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_{\text{int}} = \mathscr{H}_0 + \mathscr{H}_{\text{int}}^{\text{Yuk}} = \mathscr{H}_{\text{KG}} + \mathscr{H}_{\text{Dirac}} + \mathscr{H}_{\text{int}}^{\text{Yuk}} .$$

Quantum electrodynamics (QED):

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}^{\text{QED}} = \mathscr{L}_{\text{Dirac}} + \mathscr{L}_{\text{Maxwell}} + \mathscr{L}_{\text{int}}^{\text{QED}}$$

$$= \left[\bar{\psi}(i\partial\!\!\!/ - m)\psi\right] + \left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] + \left[-q_c\bar{\psi}A\!\!\!/\psi\right] = \bar{\psi}(iD\!\!\!\!/ - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} ,$$

$$\mathscr{L}_{\text{int}}^{\text{QED}}(A^\mu, \bar{\psi}, \psi) \equiv -q_c A_\mu j^\mu_{\text{Dirac}} = -q_c A_\mu \bar{\psi}\gamma^\mu\psi = -q_c\bar{\psi}A\!\!\!/\psi , \quad D_\mu \equiv \partial_\mu + iq_c A_\mu ,$$

QED is a $U(1)$ gauge theory

(See Chapter 7, Secs 7.4, 7.5 and 9.1)

QED as a gauge theory: The Dirac action is invariant under a *global phase transformation*, $\psi(x) \to e^{i\alpha}\psi(x)$. Using Noether's theorem leads to the conserved fermion current $j^{\mu}_{\text{Dirac}} = :\bar{\psi}\gamma^{\mu}\psi:$. as we saw earlier. Combine a *local phase transformation* $\psi(x) \to e^{i\alpha(x)}\psi(x)$ of the fermion field and a simultaneous *gauge transformation of the photon field* (no change to $\mathbf{E}$ and $\mathbf{B}$),

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)\,, \quad \bar{\psi}(x) \to \bar{\psi}'(x) = e^{-i\alpha(x)}\bar{\psi}(x)\,,$$
$$A_{\mu}(x) \to A'_{\mu}(x) = A_{\mu}(x) - (1/q_c)\partial_{\mu}\alpha(x)\,.$$

$\leftarrow$ gauge transformation in QED

We refer to these two combined transformations as a gauge transformation of the theory. We note that

$\bar{\Psi}'(i\not{D}' - m_f)\Psi' = \bar{\Psi}(i\not{D} - m_f)\Psi$ and $F'_{\mu\nu}F'^{\mu\nu} = F_{\mu\nu}F^{\mu\nu}$

and so we say that *QED is gauge invariant*. Since $e^{i\alpha}$ is an element of $U(1)$ we say that QED is a $U(1)$ gauge theory and hence abelian. Since $U(1)$ is abelian we say that QED is an abelian gauge theory.

Nonabelian gauge theories: This illustrates how to produce a gauge theory from a non-gauge theory: (i) turn a global phase invariance into a local one; (ii) introduce gauge fields and a corresponding covariant derivative $D_{\mu} = \partial_{\mu} + igA_{\mu}$; (iii) the resulting theory is invariant under the combined gauge transformation for ψ and A^{μ}.

If the global phase invariance is $e^{i\vec{\omega}\cdot\vec{T}}$ with T^a the matrix generators of some nonabelian Lie group, then we will arrive at a *nonabelian gauge theory* using this construction, e.g, *quantum chromodynamics* is the nonabelian gauge theory corresponding tp $SU(3)$ associated with quark and gluon color with the T^a as 3×3 matrix representation of $SU(3)$ matrices. Then $A_{\mu} \equiv A^a_{\mu}T^a$ and $\psi = (\psi_1, \psi_2, \psi_3) = (\psi_r, \psi_b, \psi_g)$.
See Chapter 9, Sec 9.1 and Georgio's lectures for details.

Feynman rules

(See Chapter 7, Sec 7.6)

External lines:

(scalar in)	$= 1$	(scalar out)	$= 1 \quad (7.6.3)$
(fermion in)	$= u^s(p)$	(fermion out)	$= \bar{u}^s(p)$
(antifermion in)	$= \bar{v}^s(p)$	(antifermion out)	$= v^s(p)$
(photon in)	$= \epsilon^\mu(p,\lambda)$	(photon out)	$= \epsilon^\mu(p,\lambda)^*$
(charged scalar in)	$= 1$	(charged scalar out)	$= 1$
(anti-charged scalar in)	$= 1$	(anti-charged scalar out)	$= 1\,.$

Internal lines:

(neutral, charged scalar)
$$= \frac{i}{p^2 - m^2 + i\epsilon} = D_0(p) \qquad (7.6.4)$$

(fermion) $\alpha \xrightarrow{p} \beta$
$$= \frac{i(\not{p} + m)_{\beta\alpha}}{p^2 - m^2 + i\epsilon} = S_0(p)_{\beta\alpha}$$

(photon - in R_ξ gauge)
$$= \frac{i}{p^2 + i\epsilon}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right] = D_0^{\mu\nu}(p)$$

(neutral, charged massive vector)
$$= \frac{i}{p^2 - m^2 + i\epsilon}\left[-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\right] = \Delta_0^{\mu\nu}(p)\,.$$

Feynman rules

(See Chapter 7, Sec 7.4.2 and 7.6.1)

Relative signs for Feynman diagrams: The set of all contractions leads to the set of all topologically distinct Feynman diagrams, *all topologically distinct diagrams need to be included*.
If two diagrams are topologically identical up to the exchange of two external boson lines or two external fermions lines then there is *a relative plus sign or relative minus sign* between the diagrams respectively. This is true independent of whether the lines are both in the initial state, both in the final state or one of each.

Identical particles in the final state: For n identical final-state particles include a factor of $1/n!$ in $d\Pi_n^{\text{LIPS}}$ so as not to overcount final states.

Divide diagrams by their symmetry factor: The symmetry factor S for a Feynman diagram is the number of symmetries that the diagram possesses under propagator and vertex exchanges. For each Feynman diagram and divide the contribution by its symmetry factor.

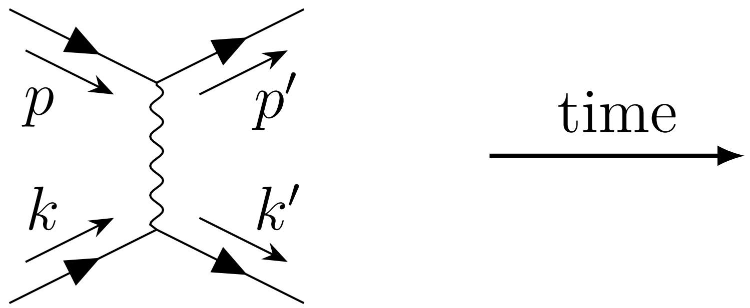
Add a minus sign for fermion loops: When a Feynman diagram contains a fermion loop it comes with a minus sign.

For detailed explanations of all these rules see Chapter 7.

Example QED tree level (no loop) calculations

(See Chapter 7, Sec 7.6.3)

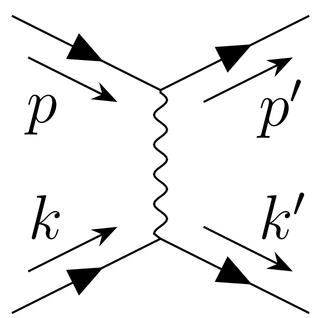
$$i\mathcal{M}_{f_1 f_2 \to f_1 f_2} = $$



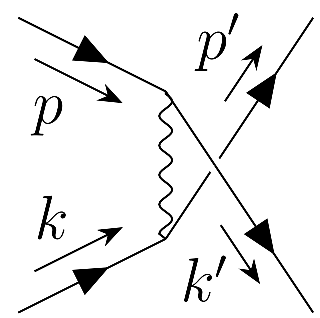
$$(7.6.43)$$

$$= (-iq_1)\bar{u}_1^{s'}(p')\gamma^\mu u_1^s(p)\frac{i\left[-g_{\mu\nu}+(1-\xi)(q_\mu q_\nu/q^2)\right]}{q^2}(-iq_2)\bar{u}_2^{r'}(k')\gamma^\nu u_2^r(k)\,,$$

$$i\mathcal{M}_{ff\to ff} = $$



$-$

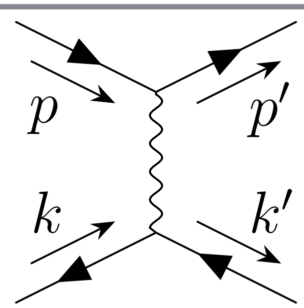


Easiest to use Feynman gauge now $\xi = 1$.

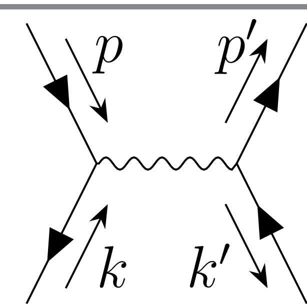
$$(7.6.45)$$

$$= i(-iq_c)^2\left\{\frac{\bar{u}^{s'}(p')\gamma^\mu u^s(p)[-g_{\mu\nu}]\bar{u}^{r'}(k')\gamma^\mu u^r(k)}{t} - \frac{\bar{u}^{s'}(p')\gamma^\mu u^r(k)[-g_{\mu\nu}]\bar{u}^{r'}(k')\gamma^\nu u^s(p)}{u}\right\},$$

$$i\mathcal{M}_{f\bar{f}\to f\bar{f}} = $$



$-$



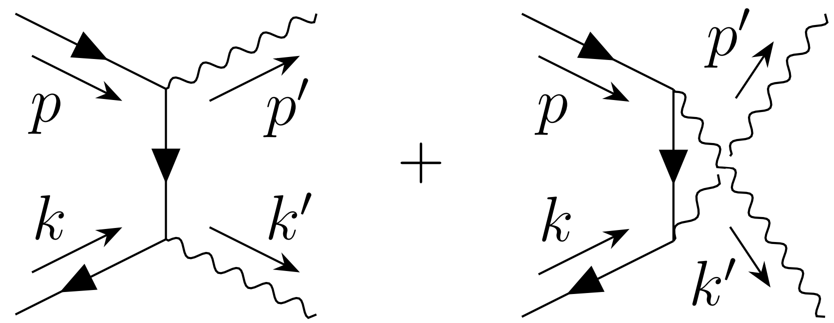
$$(7.6.46)$$

$$= i(-iq_c)^2\left\{\frac{\bar{u}^{s'}(p')\gamma^\mu u^s(p)[-g_{\mu\nu}]\bar{v}^r(k)\gamma^\nu v^{r'}(k')}{t} - \frac{\bar{u}^{s'}(p')\gamma^\mu v^{r'}(k')[-g_{\mu\nu}]\bar{v}^r(k)\gamma^\nu u^s(p)}{s}\right\},$$

Example QED tree level (no loop) calculations

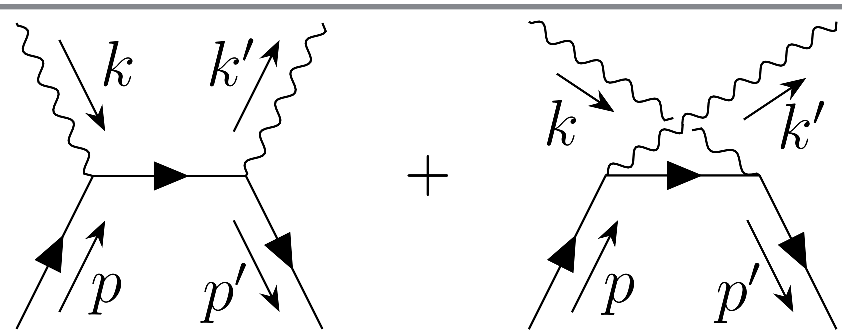
(See Chapter 7, Sec 7.6.3)

$$i\mathcal{M}_{f\bar{f}\to\gamma\gamma} = $$



$$= i(-iq_c)^2 \epsilon^\mu(p',\lambda')^* \epsilon^\nu(k',\kappa')^* \bar{v}^r(k) \left\{ \frac{\gamma_\nu[(\not{p}-\not{p}')+m]\gamma_\mu}{t-m^2} + \frac{\gamma_\mu[(\not{p}-\not{k}')+m]\gamma_\nu}{u-m^2} \right\} u^s(p), \tag{7.6.47}$$

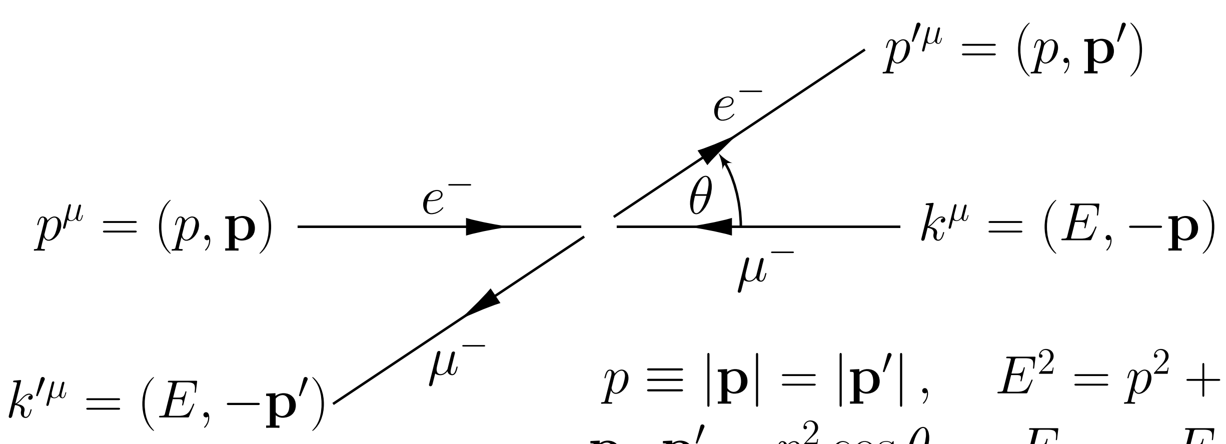
$$i\mathcal{M}_{f\gamma\to f\gamma} = $$



$$= i(-iq_c)^2 \epsilon^\nu(k',\kappa')^* \bar{u}^{s'}(p') \left\{ \frac{\gamma_\nu[(\not{p}+\not{k})+m]\gamma_\mu}{s-m^2} + \frac{\gamma_\mu[(\not{p}-\not{k}')+m]\gamma_\nu}{u-m^2} \right\} u^s(p)\epsilon^\mu(k,\lambda). \tag{7.6.48}$$

Example QED cross-section calculation

(See Chapter 7, Sec 7.6.5)



$$p \equiv |\mathbf{p}| = |\mathbf{p}'| \,, \quad E^2 = p^2 + m_\mu^2$$
$$\mathbf{p} \cdot \mathbf{p}' = p^2 \cos\theta \,, \quad E_{\rm cm} = E + p. \qquad (7.6.76)$$

Differential cross-section for $e^-\mu^- \to e^-\mu^-$: Only the first diagram in Eq. (7.6.45) will contribute since e^- and μ^- are distinguishable, which gives

$$i\mathcal{M} = ie^2 \, \frac{\bar{u}^{s'}(p')\gamma^\mu u^s(p)\bar{u}^{r'}(k')\gamma_\mu u^r(k)}{t} \,, \qquad (7.6.74)$$

Using Eq. (7.6.61) for the unpolarized cross-section the analog of Eq. (7.6.65) is

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{4}\textstyle\sum_{r,s,r',s'}|\mathcal{M}|^2 = \frac{e^4}{4t^2}\,\mathrm{tr}[\gamma^\mu(\not{k}+m_\mu)\gamma^\nu(\not{k}'+m_\mu)]\,\mathrm{tr}[\gamma_\mu(\not{p}+m_e)\gamma_\nu(\not{p}'+m_e)] \\ &= \frac{8e^4}{t^2}\left[p\cdot k' p'\cdot k + p\cdot k p'\cdot k' - m_\mu^2 p\cdot p' - m_e^2 k\cdot k' + 2m_e^2 m_\mu^2\right] \\ &= \frac{2e^4}{t^2}\left[u^2 + s^2 + 4t(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2\right] \,, \qquad (7.6.75) \end{aligned}$$

where we have p, p' for e^- and k, k' for μ^-. The first line above is equivalent to the first line of Eq. (7.6.65) with the replacements $(p, k; p', k') \to (p, -p'; k', -k)$.