Lecture 4: Quantum field theory for fermions, quantized electromagnetic field and gauge fixing

Fermi-Dirac statistics

(See Chapter 6, Secs 6.3.1)

Fermi-Dirac statistics: Since fermions must *obey fermi-Dirac statistics* and hence the Pauli exclusion principle we need to build states that are *antisymmetric under pairwise particle exchange*, c.f., symmetric states under pairwise exchange for bosons.

Toy example: Consider a system with only N possible basis states and a single fermion species. We can put at most one such fermion in each basis state. Denote the annihilation and creation operators for the single-fermion basis state $|b_i\rangle$ as $\hat{b}_i$ and $\hat{b}_i^\dagger$ respectively with the necessary properties

$$\hat{b}_i^\dagger|0\rangle = |b_i\rangle\,, \quad \hat{b}_i^\dagger|b_i\rangle = 0\,, \quad \hat{b}_i|b_i\rangle = |0\rangle\,, \quad \hat{b}_i|0\rangle = 0\,,$$

which we can summarize in terms of the i^{th}basis state occupation number $n_i = 0{,}1$ as

$$\hat{b}_i^\dagger|n_i\rangle = (1-n_i)|n_i+1\rangle\,, \qquad \hat{b}_i|n_i\rangle = n_i|n_i-1\rangle$$

So if the state $|b_i\rangle$ is unoccupied then $\hat{b}_i^\dagger$ creates a fermion in that state and $\hat{b}_i$ acting on the state gives zero, whereas if $|b_i\rangle$ is occupied then $\hat{b}_i^\dagger$ acting on the state gives zero and $\hat{b}_i$ annihilates a fermion from that state. We also note that $\hat{N}_i = \hat{b}_i^\dagger\hat{b}_i$ is the number operator for the i^{th} basis state and that $\{\hat{b}_i, \hat{b}_i^\dagger\}|n_i\rangle = (\hat{b}_i\hat{b}_i^\dagger + \hat{b}_i^\dagger\hat{b}_i)|n_i\rangle = |n_i\rangle$, since

$$\hat{N}_i|n_i\rangle = \hat{b}_i^\dagger\hat{b}_i|n_i\rangle = n_i\hat{b}_i^\dagger|n_i-1\rangle = n_i[1-(n_i-1)]|n_i\rangle = n_i|n_i\rangle\,,$$

$$\hat{b}_i\hat{b}_i^\dagger|n_i\rangle = (1-n_i)\hat{b}_i|n_i+1\rangle = (1-n_i)(n_i+1)|n_i\rangle = (1-n_i)|n_i\rangle\,,$$

$$\{\hat{b}_i, \hat{b}_i^\dagger\}|n_i\rangle = [n_i + (1-n_i)]|n_i\rangle = |n_i\rangle\,.$$

Define now $\hat{b}_{i_1}^\dagger\hat{b}_{i_2}^\dagger\cdots\hat{b}_{i_n}^\dagger|0\rangle \equiv |b_{i_1}b_{i_2}\cdots b_{i_n}\rangle$, which vanishes if any two $b_i^\dagger$ are the same. We require the state to be antisymmetric and so the pairwise exchange of two different $b_i^\dagger$ must also give a minus sign.

Fock space for Dirac fermions

(See Chapter 6, Secs 6.3.1 and 6.3.4)

This means that we must have $\hat{b}_i^\dagger \hat{b}_j^\dagger = -\hat{b}_j^\dagger \hat{b}_i^\dagger \implies \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = 0$ if $i \neq j$.

For consistency we also then require that $\{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i, \hat{b}_j^\dagger\} = 0$ for $i \neq j$. So we have arrived at the anticommutation relations for fermion annihilation and creation operators for our toy model,

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \delta_{ij} \quad \text{and} \quad \{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = 0 \, .$$

We will generalize these results when constructing our quantum field theory for Dirac fermions. The set of all possible states $\hat{b}_{i_1}^\dagger \hat{b}_{i_2}^\dagger \cdots \hat{b}_{i_n}^\dagger |0\rangle \equiv |b_{i_1} b_{i_2} \cdots b_{i_n}\rangle$ then form a basis of fermion the Fock space and every state in the Fock space is some linear combination of these basis states. Obviously the system can never contain more than N fermions, since there is are only N orthogonal single-fermion states available.

Fock space for fermions and anti fermions: Denote the annihilation and creation operators as $\hat{b}_j$ and $\hat{b}_j^\dagger$ for fermions and $\hat{d}_j$ and $\hat{d}_j^\dagger$ for antifermions respectively. Let $\hat{N}_{bj} = \hat{b}_j^\dagger \hat{b}_j$ and $\hat{N}_{dj} = \hat{d}_j^\dagger \hat{d}_j$ be the number operators measuring the number of fermions and antifermions respectively in the state j with energy E_j. Then the normal-ordered Hamiltonian for free particles must have the form

$$:\hat{H}: \; \equiv \; \textstyle\sum_{j=1}^N E_j(\hat{N}_{bj} + \hat{N}_{dj}) = \sum_{j=1}^N E_j(\hat{b}_j^\dagger \hat{b}_j + \hat{d}_j^\dagger \hat{d}_j) \, ,$$

where the annihilation operators annihilate the vacuum so that an empty state has no energy,

$$\hat{b}_j |0\rangle = \hat{d}_j |0\rangle = 0.$$

This is analogous to the discussion of the charged scalar field. The anticommutation relations are

$$\{\hat{b}_i, \hat{b}_j^\dagger\} = \{\hat{d}_i, \hat{d}_j^\dagger\} = \delta_{ij} \, , \quad \{\hat{b}_i, \hat{b}_j\} = \{\hat{b}_i^\dagger, \hat{b}_j^\dagger\} = \{\hat{d}_i, \hat{d}_j\} = \{\hat{d}_i^\dagger, \hat{d}_j^\dagger\} = 0 \, ,$$

$$\{\hat{b}_i, \hat{d}_j\} = \{\hat{b}_i^\dagger, \hat{d}_j\} = \{\hat{b}_i, \hat{d}_j^\dagger\} = \{\hat{b}_i^\dagger, \hat{d}_j^\dagger\} = 0 \, .$$

Dirac Hamiltonian

(See Chapter 6, Sec 6.3.4)

Fock space for Dirac fermions: For Dirac fermions the single particle basis states are the plane wave solutions of the Dirac equation, which are the normal modes of the Dirac fermion system. This corresponds to $j \to (s, \mathbf{p})$, where $s = \pm(1/2)$ is the spin state and $\mathbf{p}$ is the three-momentum of the state. For a theory of free relativistic fermions there must be a corresponding normal-ordered Hamiltonian operator such that

$$\hat{H} \equiv \int (d^3p/(2\pi)^3) \sum_{s=\pm(1/2)} E_{\mathbf{p}}(\hat{b}_{\mathbf{p}}^{s\dagger}\hat{b}_{\mathbf{p}}^{s} + \hat{d}_{\mathbf{p}}^{s\dagger}\hat{d}_{\mathbf{p}}^{s}) \equiv \int (d^3p/(2\pi)^3) \sum_{s=\pm(1/2)} E_{\mathbf{p}}(\hat{N}_{b\mathbf{p}s} + \hat{N}_{d\mathbf{p}s})$$

with $E_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2} > 0$ and where the anticommutation relations are

$$\{\hat{b}_{\mathbf{p}}^{s}, \hat{b}_{\mathbf{p}'}^{s'\dagger}\} = \{\hat{d}_{\mathbf{p}}^{s}, \hat{d}_{\mathbf{p}'}^{s'\dagger}\} = \delta^{ss'}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'),$$

$$\{\hat{b}_{\mathbf{p}}^{s}, \hat{b}_{\mathbf{p}'}^{s'}\} = \{\hat{b}_{\mathbf{p}}^{s\dagger}, \hat{b}_{\mathbf{p}'}^{s'\dagger}\} = \{\hat{d}_{\mathbf{p}}^{s}, \hat{d}_{\mathbf{p}'}^{s'}\} = \{\hat{d}_{\mathbf{p}}^{s\dagger}, \hat{d}_{\mathbf{p}'}^{s'\dagger}\} = 0,$$

$$\{\hat{b}_{\mathbf{p}}^{s}, \hat{d}_{\mathbf{p}'}^{s'}\} = \{\hat{b}_{\mathbf{p}}^{s\dagger}, \hat{d}_{\mathbf{p}'}^{s'}\} = \{\hat{b}_{\mathbf{p}}^{s}, \hat{d}_{\mathbf{p}'}^{s'\dagger}\} = \{\hat{b}_{\mathbf{p}}^{s\dagger}, \hat{d}_{\mathbf{p}'}^{s'\dagger}\} = 0.$$

We not that the total energy is the sum of the energies of the fermions and the antifermions.

We define the one-fermion and one-antifermion states created by $\hat{b}_{\mathbf{p}}^{s\dagger}$ and $\hat{d}_{\mathbf{p}}^{s\dagger}$ respectively as

$$|f; \mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, b_{\mathbf{p}}^{s\dagger}|0\rangle \quad \text{and} \quad |\bar{f}; \mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}}\, d_{\mathbf{p}}^{s\dagger}|0\rangle,$$

where the normalization is chosen such that the bra-kets of such states are Lorentz invariant. We find

$$\langle f; \mathbf{p}, s | f; \mathbf{p}', s'\rangle = \langle \bar{f}; \mathbf{p}, s | \bar{f}; \mathbf{p}', s'\rangle = 2E_{\mathbf{p}}\delta^{ss'}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'),$$

$$\langle \bar{f}; \mathbf{p}, s | f; \mathbf{p}', s'\rangle = \langle f; \mathbf{p}, s | \bar{f}; \mathbf{p}', s'\rangle = 0.$$

The results immediately follow from the anti commutation relations.

Dirac field operators

(See Chapter 6, Secs 6.3.1 and 6.3.4)

We know that $u^s(p)e^{-ip\cdot x}$ and $v^s(p)e^{ip\cdot x}$ are the plane wave solutions (normal modes) of the Dirac equation $(i\partial\!\!\!/ - m)u^s(p)e^{-ip\cdot x} = (i\partial\!\!\!/ - m)v^s(p)e^{ip\cdot x} = 0$. We define the Schrödinger-picture operators

$$\hat{\psi}(\mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^3}\left(1/\sqrt{2E_\mathbf{p}}\right)\sum_{s=\pm(1/2)}\left[\hat{b}^s_\mathbf{p}u^s(p)e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{d}^{s\dagger}_\mathbf{p}v^s(p)e^{-i\mathbf{p}\cdot\mathbf{x}}\right],$$

$$\hat{\psi}^\dagger(\mathbf{x}) \equiv \int \frac{d^3p}{(2\pi)^3}\left(1/\sqrt{2E_\mathbf{p}}\right)\sum_{s=\pm(1/2)}\left[\hat{b}^{s\dagger}_\mathbf{p}u^{s\dagger}(p)e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{d}^{s}_\mathbf{p}v^{s\dagger}(p)e^{i\mathbf{p}\cdot\mathbf{x}}\right],$$

where the spinor index $\alpha = 1,2,3,4$ is suppressed for brevity on $u^s(p)$, $v^s(p)$, $\hat{\psi}$ and its hermitian conjugate $\hat{\psi}^\dagger$. These definitions and the anti commutation relations lead to

$$\hat{H} \equiv H[\hat{b}^\dagger, \hat{d}^\dagger, \hat{b}, \hat{d}] \equiv \int d^3x\, \mathscr{H}(\hat{b}^\dagger, \hat{d}^\dagger, \hat{b}, \hat{d}) \equiv \int d^3x \; :\hat{\bar{\psi}}(\mathbf{x})(-i\gamma\cdot\nabla + m)\hat{\psi}(\mathbf{x}):$$

$$= \int (d^3p/(2\pi)^3) \sum_s E_\mathbf{p} :(\hat{b}^{s\dagger}_\mathbf{p}\hat{b}^s_\mathbf{p} - \hat{d}^s_\mathbf{p}\hat{d}^{s\dagger}_\mathbf{p}): \;=\; \int (d^3p/(2\pi)^3) \sum_s E_\mathbf{p}(\hat{b}^{s\dagger}_\mathbf{p}\hat{b}^s_\mathbf{p} + \hat{d}^{s\dagger}_\mathbf{p}\hat{d}^s_\mathbf{p}),$$

where fermion normal ordering brings a -ve sign. We identify the normal-ordered Hamiltonian density as

$$\mathscr{H} \;\equiv\; :\hat{\bar{\psi}}(\mathbf{x})(-i\gamma\cdot\nabla + m)\hat{\psi}(\mathbf{x}): .$$

It also follows that we then have

$$\{\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}^\dagger_\beta(\mathbf{y})\} = \delta_{\alpha\beta}\,\delta^3(\mathbf{x}-\mathbf{y}) \qquad \text{and} \qquad \{\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}_\beta(\mathbf{y})\} = \{\hat{\psi}^\dagger_\alpha(\mathbf{x}), \hat{\psi}^\dagger_\beta(\mathbf{y})\} = 0,$$

which are the *canonical anticommutation relations for the fermion field*.

The Heisenberg picture operators are then (choosing $t_0 = 0$ for convenience)

$$\hat{\psi}(x) \equiv e^{i\hat{H}t}\hat{\psi}(\mathbf{x})e^{-i\hat{H}t} = \int (d^3p/(2\pi)^3)\left(1/\sqrt{2E_\mathbf{p}}\right)\sum_s\left[\hat{b}^s_\mathbf{p}u^s(p)e^{-ip\cdot x} + \hat{d}^{s\dagger}_\mathbf{p}v^s(p)e^{ip\cdot x}\right],$$

$$\hat{\psi}^\dagger(x) \equiv e^{i\hat{H}t}\hat{\psi}^\dagger(\mathbf{x})e^{-i\hat{H}t} = \int (d^3p/(2\pi)^3)\left(1/\sqrt{2E_\mathbf{p}}\right)\sum_s\left[\hat{b}^{s\dagger}_\mathbf{p}u^{s\dagger}(p)e^{ip\cdot x} + \hat{d}^s_\mathbf{p}v^{s\dagger}(p)e^{-ip\cdot x}\right].$$

Canonical quantization of the Dirac field

(See Chapter 6, Secs 6.3.6)

Simple canonical quantization argument: Consider the Dirac action

$$S[\bar\psi,\psi] = \int dt\, L = \int d^4x\, \mathscr{L} = \int d^4x\, \bar\psi(x)(i\not\partial - m)\psi(x)$$

where $\psi(x)$ is a four-component column vector with components $\psi_\alpha(x) \in \mathbb{C}$, where $\psi^\dagger(x)$ is the complex conjugate row vector with components $\psi^*_\alpha(x)$, and where $\bar\psi_\alpha(x) \equiv \psi^\dagger(x)\gamma^0$ is the Dirac adjoint spinor. Similarly to the complex scalar field case, we can then express Hamilton's principle for the Dirac action as

$$\frac{\delta S[\bar\psi,\psi]}{\delta\bar\psi_\alpha(x)} = \frac{\delta S[\bar\psi,\psi]}{\delta\psi_\alpha(x)} = 0$$

which leads to the recovery of the Dirac equation as the equations of motion,

$$(i\not\partial - m)\psi(x) = \bar\psi(x)(-i\overleftarrow{\not\partial} - m) = 0\,.$$

The canonical momentum densities conjugate to ψ_α and $\bar\psi_\alpha$ are respectively,

$$\pi_\alpha(x) = \delta L/\delta\dot\psi_\alpha(x) = \partial\mathscr{L}/\partial\dot\psi_\alpha(x) = i\psi^\dagger_\alpha(x) = i(\bar\psi(x)\gamma^0)_\alpha\,, \qquad \bar\pi_\alpha(x) = \delta L/\delta\dot{\bar\psi}_\alpha(x) = \partial\mathscr{L}/\partial\dot{\bar\psi}_\alpha(x) = 0\,.$$

We see that we have a problem since $\bar\pi = 0$ means that we have a singular system and we should use the Dirac-Bergmann algorithm at this point. But let us press on anyway and form the Hamiltonian

$$H = \int d^3x \sum_\alpha [\dot\psi_\alpha\pi_\alpha + \dot{\bar\psi}_\alpha\bar\pi_\alpha] - L = \int d^3x\, [\sum_\alpha(\dot\psi_\alpha\pi_\alpha + \dot{\bar\psi}_\alpha\bar\pi_\alpha) - \mathscr{L}]$$
$$= \int d^3x\, [i\bar\psi\gamma^0\partial_0\psi - \bar\psi(i\not\partial - m)\psi] = \int d^3x\, \bar\psi[-i\gamma\cdot\nabla + m]\psi = \int d^3x\, \mathscr{H}\,.$$

We define in the usual way $\pi^\mu_\alpha \equiv \partial\mathscr{L}/\partial(\partial_\mu\psi_\alpha) = i(\bar\psi\gamma^\mu)_\alpha$ and $\bar\pi^\mu_\alpha \equiv \partial\mathscr{L}/\partial(\partial_\mu\bar\psi_\alpha) = 0\,.$

Spacetime translational invariance leads to the conserved stress-energy tensor

$$T^\mu{}_\nu = \pi^\mu\,\partial_\nu\psi + \bar\pi^\mu\,\partial_\nu\bar\psi = i\bar\psi\gamma^\mu\partial_\nu\psi \quad \text{where} \quad \partial_\mu T^\mu{}_\nu = 0\,.$$

Canonical quantization of the Dirac field

(See Chapter 6, Secs 6.3.6)

The four conserved charges are the $\hat{P}^\mu$, where we find that

$$P^\nu = (H, \mathbf{P}) = \int d^3x\, T^{0\nu} = \int d^3x\, i\bar{\psi}\gamma^0\partial^\nu\psi = \int d^3x\, i\psi^\dagger\partial^\nu\psi\,.$$

Since $(i\,\partial\!\!\!/ - m)\psi = 0$ on-shell then we can write the on-shell Hamiltonian as

$$H = \int d^3x\, i\psi^\dagger\partial_0\psi = \int d^3x\, \psi^\dagger(-i\alpha\cdot\nabla + \beta m)\psi = \int d^3x\, \bar{\psi}[-i\gamma\cdot\nabla + m]\psi$$

The total conserved three-momentum is given by

$$\mathbf{P} = \int d^3x\, \psi^\dagger(-i\nabla)\psi\,.$$

The Dirac action is Lorentz invariant by construction and we have a conserved angular momentum tensor

$$M^{\rho\sigma} = \int d^3x \left[\left(x^\rho T^{0\sigma} - x^\sigma T^{0\rho}\right) + \pi_\alpha(\Sigma_{\alpha\beta})^{\rho\sigma}\psi_\beta + \bar{\pi}_\alpha(\Sigma_{\alpha\beta})^{\rho\sigma}\bar{\psi}_\beta\right]$$

$$= \int d^3x\, [\left(x^\rho i\psi^\dagger\partial^\sigma\psi - x^\sigma i\psi^\dagger\partial^\rho\psi\right) + i\psi^\dagger_\alpha(\Sigma_{\alpha\beta})^{\rho\sigma}\psi_\beta]$$

$$= \int d^3x\, i\psi^\dagger\left(x^\rho\partial^\sigma - x^\sigma\partial^\rho + frac14[\gamma^\rho,\gamma^\sigma]\right)\psi = \int d^3x\, \psi^\dagger\left(x^\rho i\partial^\sigma - x^\sigma i\partial^\rho + \Sigma^{\mu\nu}_{\text{Dirac}}\right)\psi\,,$$

where $i\Sigma^{\rho\sigma} \equiv \Sigma^{\rho\sigma}_{\text{Dirac}} = \frac{1}{2}\sigma^{\rho\sigma} = \frac{i}{4}[\gamma^\rho,\gamma^\sigma]$. The angular momentum is conserved and is given by

$$\mathbf{J} = (M^{23}, M^{31}, M^{12}) = \int d^3x\, \psi^\dagger\left(\mathbf{x}\times(-i\nabla) + \mathbf{S}\right)\psi = \int d^3x\, \psi^\dagger\left(\mathbf{x}\times(-i\nabla) + \frac{1}{2}\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}\right)\psi\,.$$

Applying Dirac correspondence principle at this point leads to the wrong outcome

$$[\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}^\dagger_\beta(\mathbf{y})] = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}) \quad \text{and} \quad [\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}_\beta(\mathbf{y})] = [\hat{\psi}^\dagger_\alpha(\mathbf{x}), \hat{\psi}^\dagger_\beta(\mathbf{y})] = 0\,.$$

The simplest thing to do is to simply replace the commutators with anticommutators because we are dealing with fermions and then hope everything works out OK, which it does.

More careful canonical argument: The above cavalier argument leads to the right outcome! It is not possible to give details here (see Sec 6.3.6). In brief, if we use the full Dirac-Bergmann algorithm and introduce anticommuting (Grassmann) classical fermion fields we correctly arrive at these same results and all symmetries, Noether currents and the Poincaré Lie algebra are preserved.

Quantum field theory for Dirac fermions

(See Chapter 6, Sec 6.3.7)

With the previous arguments we have arrived at the normal-ordered four-momentum and angular momentum tensor

$$\hat{P}^\mu = \int d^3x \; :i\hat{\psi}^\dagger(x)\partial^\mu\hat{\psi}(x): \; , \qquad \hat{M}^{\mu\nu} = \int d^3x \; :i\hat{\psi}^\dagger \left(x^\mu\partial^\nu - x^\nu\partial^\mu + \tfrac{1}{4}[\gamma^\mu, \gamma^\nu] \right) \hat{\psi}: ,$$

where for example $\hat{P}^\mu = (\hat{H}, \hat{\mathbf{P}}) = \int (d^3p/(2\pi)^3) \sum_s p^\mu (\hat{b}^{s\dagger}_{\mathbf{p}} \hat{b}^s_{\mathbf{p}} + \hat{d}^{s\dagger}_{\mathbf{p}} \hat{d}^s_{\mathbf{p}})$.

The Dirac action $S[\psi, \bar{\psi}]$ is invariant under global phase transformations since if $\psi(x) \to e^{i\alpha}\psi(x)$ then $\bar{\psi}(x) \to e^{-i\alpha}\bar{\psi}(x)$ and so $S[\psi, \bar{\psi}] \to S[\psi, \bar{\psi}]$. The corresponding Noether current can be written as

$$j^\mu = -\sum_\beta [\Phi_\beta(x)\pi^\mu_\beta(x) + \bar{\Phi}_\beta(x)\bar{\pi}^\mu_\beta(x)] = -\sum_\beta (i\psi_\beta)(-i[\bar{\psi}\gamma^\mu]_\beta) = \bar{\psi}\gamma^\mu\psi ,$$

where we have chosen the conventional sign for the current. The normal-ordered current density operator will also be conserved since it differs by a constant and so

$$\hat{j}^\mu(x) \equiv :\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x): \qquad \text{with} \qquad \partial_\mu \hat{j}^\mu = 0 \, .$$

The corresponding conserved charge operator is

$$\hat{Q} = \int d^3x \, \hat{j}^0(x) = \int d^3x \; :\hat{\psi}^\dagger(x)\hat{\psi}(x): \; = \; \int (d^3p/(2\pi)^3) \sum_s (\hat{b}^{s\dagger}_{\mathbf{p}} \hat{b}^s_{\mathbf{p}} - \hat{d}^{s\dagger}_{\mathbf{p}} \hat{d}^s_{\mathbf{p}}) \equiv \hat{N}_b - \hat{N}_d = \hat{N},$$

which is the fermion number operator minus the antifermion number operator.

We can prove (as expected) that the Poincaré Lie algebra survives in the quantum field theory

$$[\hat{P}^\mu, \hat{P}^\nu] = 0 \, , \qquad [\hat{P}^\mu, \hat{M}^{\rho\sigma}] = i(g^{\mu\rho}\hat{P}^\sigma - g^{\mu\sigma}\hat{P}^\rho) \, ,$$

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = i(g^{\nu\rho}\hat{M}^{\mu\sigma} - g^{\mu\rho}\hat{M}^{\nu\sigma} - g^{\nu\sigma}\hat{M}^{\mu\rho} + g^{\mu\sigma}\hat{M}^{\nu\rho}) \, .$$

The conserved charge operators $\hat{P}^\mu$ and $\hat{M}^{\mu\nu}$ are the generators of a unitary representation of the translations and Lorentz transformations respectively, $\hat{U}(a) = e^{i\hat{P}\cdot a}$ and $\hat{U}(\Lambda) = e^{-(i/2)\omega_{\mu\nu}\hat{M}^{\mu\nu}}$.

Fermion propagator and Grassmann algebra

(See Chapter 6, Secs 6.3.2 and 6.3.7)

Feynman fermion propagator: The fermion propagator (or Green's function) satisfies by definition

$$(i\,\partial\!\!\!/_y - m)S_F(y-x) = i\delta^4(y-x)$$

and the solution to this with the correct Feynman boundary conditions is

$$S_F(x-y) \equiv \langle 0|\,T[\hat{\psi}(x)\hat{\bar{\psi}}(y)]\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}e^{-ip\cdot(x-y)} \equiv \int \frac{d^4p}{(2\pi)^4}S_F(p)e^{-ip\cdot(x-y)}\,.$$

Grassmann algebra: It is not possible to discuss this topic in detail but the generators, a_i, of a Grassmann algebra anticommute $\{a_i, a_j\} = 0$. They can be thought of as anticommuting c-numbers. They arise because the allow a representation of fermion Fock space, i.e., we can identify a_i with an occupied fermion state i. So we can represent $|b_{i_1}b_{i_2}\ldots b_{i_n}\rangle$ with the Grassmann product $a_{i_1}a_{i_2}\ldots a_{i_n}$.

Note that even Grassmann elements commute with everything, e.g., $(a_ia_j)a_k = a_k(a_ia_j)$.

The rules of Grassmann algebra allow a representation of actions in fermion Fock space (see Sec. 6.3.2 for details). The (left) Grassmann derivative satisfies

$$\frac{\partial}{\partial a_i}1 \equiv 0\,, \quad \frac{\partial}{\partial a_i}a_j \equiv \delta_{ij}\,, \quad \frac{\partial}{\partial a_i}a_j \equiv -\,a_j\frac{\partial}{\partial a_i} \text{ if } i \neq j\,, \quad \frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j} = -\,\frac{\partial}{\partial a_j}\frac{\partial}{\partial a_i}$$

and the definition of Grassmann integration is

$$\int da_i\, 1 \equiv 0\,, \quad \int da_i\, a_j \equiv -\int a_j\, da_i = \delta_{ij}\,, \quad \left\{a_i, da_j\right\} \equiv \left\{da_i, da_j\right\} \equiv 0\,,$$

which has similarities with Grassmann differentiation.

Fermion generating functional

(See Chapter 6, Secs 6.3.8)

Fermion spectral function: The spectral function for fermions can be expressed in terms of a Grassman functional integral over both the Grassman-valued field ψ and the Grassmann-valued field $\bar{\psi}$. The fermion action $S[\bar{\psi},\psi]$ is an even Grassman element and so commutes with everything. The spectral function can be shown to be

$$F(t''-t') = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi]}, \qquad \text{where} \qquad S[\bar{\psi},\psi] = \int_{t'}^{t''} dt \int d^3x\, \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x)\,.$$

We can generalize this by adding source terms $\eta(x)$ and $\bar{\eta}(x)$ which are *also Grassmann-valued* and so anticommuting.

$$F^{\bar{\eta}\eta}(t'',t') = \mathrm{tr}\{\hat{U}^{\bar{\eta}\eta}(t'',t')\} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi,\bar{\eta},\eta]}, \qquad \text{where}$$

$$S[\bar{\psi},\psi,\bar{\eta},\eta] \equiv S[\bar{\psi},\psi] + \int d^4x\,[\bar{\eta}\psi + \bar{\psi}\eta] = \int_{t'}^{t''} d^4x\,\left[\mathcal{L} + \bar{\eta}\psi + \bar{\psi}\eta\right]\,.$$

Fermion generating functional: Defining the generating functional in the usual way we have

$$Z[\bar{\eta},\eta] \equiv \lim_{T\to\infty(1-i\epsilon)} \frac{F^{\bar{\eta}\eta}(T,-T)}{F(T,-T)} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\{\hat{U}^{\bar{\eta}\eta}(T,-T)\}}{\mathrm{tr}\{\hat{U}(T,-T)\}} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\left[Te^{-i\int_{-T}^{T} d^4x\,[\hat{\mathcal{H}}-\bar{\eta}\hat{\psi}-\hat{\bar{\psi}}\eta]}\right]}{\mathrm{tr}\left[Te^{-i\int_{-T}^{T} d^4x\,\hat{\mathcal{H}}}\right]}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,Te^{i\int_{-T}^{T} d^4x\,[\bar{\eta}\hat{\psi}_I+\hat{\bar{\psi}}_I\eta]}\,|0\rangle}{\langle 0|0\rangle} = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi,\bar{\eta},\eta]}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi]}}\,.$$

For a free fermion field we can write $S[\bar{\psi},\psi] = \int d^4x\, d^4y\, \bar{\psi}(x)\left[(i\partial\!\!\!/^x - m)\delta^4(x-y)\right]\psi(y)$ and perform the fermion functional integral exactly to arrive at

$$Z[\bar{\eta},\eta] \equiv \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi]+i\int d^4x\,[\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)]}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS[\bar{\psi},\psi]}} = \exp\left\{-\int d^4x\, d^4y\, \bar{\eta}(x)S_F(x-y)\eta(y)\right\}\,.$$

Fermion generating functional

(See Chapter 6, Secs 6.3.8)

Using these results we arrive at

$$\langle 0 | T\hat{\psi}(x)\hat{\bar{\psi}}(y) | 0 \rangle = \frac{\delta^2}{\delta\bar{\eta}(x)\delta\eta(y)} Z[\bar{\eta},\eta]\Bigg|_{\bar{\eta}=\eta=0} = \frac{\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\ \psi(x)\bar{\psi}(x) e^{iS[\bar{\psi},\psi]}}{\int \mathscr{D}\bar{\psi}\mathscr{D}\psi\ e^{iS[\bar{\psi},\psi]}} = S_{\mathrm{F}}(x-y)\ ,$$

which recovers the earlier result that $\langle 0 | T\hat{\psi}(x)\hat{\bar{\psi}}(y) | 0 \rangle = S_{\mathrm{F}}(x-y)$. The general result is

$$\langle 0 | T\hat{\psi}(x_1)\ldots\hat{\psi}(x_n)\hat{\bar{\psi}}(y_1)\ldots\hat{\bar{\psi}}(y_m) | 0 \rangle = (-i)^m i^n \frac{\delta^{m+n}}{\delta\bar{\eta}(x_1)\ldots\delta\bar{\eta}(x_m)\delta\eta(y_1)\ldots\delta\eta(y_n)} Z[\bar{\eta},\eta]\Bigg|_{\bar{\eta}=\eta=0}$$

$$= \delta_{mn}\Sigma_{k_1,\ldots,k_m}\epsilon^{k_1,\ldots,k_m} S_F(x_1-y_{k_1})S_F(x_2-y_{k_2})\ldots S_F(x_m-y_{k_m})\ .$$

where the factor $\epsilon^{k_1,\ldots,k_m}$ antisymmetrizes the product or propagators appropriately under pairwise particle exchange as it should. Thus the *fermion version of Wick's theorem* includes the $\epsilon^{k_1,\ldots,k_m}$. For further detail see Sec 6.3.8.

This brings us to the end of our quick survey of the quantum field theory of Dirac fermions.

Canonical quantization of the electromagnetic field

(See Chapter 6, Secs 6.4.1)

Canonical quantization of the electromagnetic field: A careful construction of the Hamiltonian formulation of electromagnetism as carried out in Chapter 3, Sec. 3.3.2. This was necessary because as we observed earlier electromagnetism is a singular system. Using the Coulomb gauge the construction of the relevant Dirac brackets was made and two constraints emerged that reduced the four degrees of freedom in A^μ to the two degrees of freedom of the electromagnetic field. It was shown that the appropriate canonical quantization relations are then

$$[\hat{E}^i(x), \hat{E}^j(y)]_{x^0=y^0} = [\hat{A}^i(x), \hat{A}^j(y)]_{x^0=y^0} = 0\,, \quad \text{and} \quad [\hat{A}^j(y), \hat{E}^i(x)]_{x^0=y^0} = -\,i\delta^{\rm tr}_{ij}(\mathbf{x}-\mathbf{y})\,,$$

where the transverse delta function is defined as

$$\delta^{\rm tr}_{ij}(\mathbf{x}-\mathbf{y}) \equiv \int [d^3k/(2\pi)^3]\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \left[\delta_{ij} - (k^i k^j/\mathbf{k}^2)\right].$$

The Hamiltonian operator for the system is

$$\hat{H} = \int d^3x \left[\tfrac{1}{2}(\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2) - (\nabla\cdot\hat{\mathbf{E}} - j^0)A_0 - \mathbf{j}\cdot\hat{\mathbf{A}}\right],$$

where the term containing $\nabla\cdot\hat{\mathbf{E}} - j^0 = \nabla\cdot\hat{\mathbf{E}} - \rho$ vanishes when acting on states in the physical subspace, i.e., the physical subspace is the null space of the operator $\nabla\cdot\hat{\mathbf{E}} - j^0$ and of the Coulomb gauge operator $\nabla\cdot\hat{\mathbf{A}}$. This ensures that Coulomb's law and the gauge fixing conditions are always satisfied, $\nabla\cdot\mathbf{E} = \rho$ and $\nabla\cdot\mathbf{A} = 0$. We can explicitly verify that this canonical quantization reproduces Maxwell's equations at the operator level as it should,

Faraday's law: $\nabla\times\hat{\mathbf{E}} = -\,\dot{\hat{\mathbf{B}}}$, Gauss' magnetism law: $\nabla\cdot\hat{\mathbf{B}} = 0$,

Gauss' law: $\nabla\cdot\hat{\mathbf{E}} = \rho$, Ampere's law: $\nabla\times\hat{\mathbf{B}} - \mathbf{j} = \dot{\hat{\mathbf{E}}}$.

Canonical quantization of the electromagnetic field

(See Chapter 6, Secs 6.4.1)

Fock space for photons: Since photons are bosons the commutation relations for the annihilation and creation operators are

$$[\hat{a}^{\lambda}_{\mathbf{k}}, \hat{a}^{\lambda'}_{\mathbf{k}'}] = [\hat{a}^{\lambda\dagger}_{\mathbf{k}}, \hat{a}^{\lambda'\dagger}_{\mathbf{k}'}] = 0 \quad \text{and} \quad [\hat{a}^{\lambda}_{\mathbf{k}}, \hat{a}^{\lambda'\dagger}_{\mathbf{k}'}] = (2\pi)^3 \delta^{\lambda\lambda'} \delta^3(\mathbf{k} - \mathbf{k}') ,$$

where $\lambda = 1{,}2$ label the polarization state of the photon state.

The normal-ordered Hamiltonian and number operators are then

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1}^{2} \omega_{\mathbf{k}} \hat{a}^{\lambda\dagger}_{\mathbf{k}} \hat{a}^{\lambda}_{\mathbf{k}} = \int d^3k \sum_{\lambda=1}^{2} \omega_{\mathbf{k}} \hat{N}^{\lambda}_{\mathbf{k}} = \int d^3k \sum_{\lambda=1}^{2} E_{\mathbf{k}} \hat{N}_{\mathbf{k}} ,$$

$$\hat{N}^{\lambda}_{\mathbf{k}} \equiv \frac{1}{(2\pi)^3} \hat{a}^{\lambda\dagger}_{\mathbf{k}} \hat{a}^{\lambda}_{\mathbf{k}} \quad \text{and} \quad \hat{N} \equiv \int d^3k \sum_{\lambda=1}^{2} \hat{N}^{\lambda}_{\mathbf{k}} .$$

Here $\omega_{\mathbf{k}} \equiv E_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$ is the energy of a photon with three-momentum $\mathbf{k}$ and polarization $\lambda = 1{,}2$ and $\hat{N}^{\lambda}_{\mathbf{k}}$ is the corresponding occupation number density. The operator $\hat{N}$ measures the total number of photons. Since $\hat{a}^{\lambda\dagger}_{\mathbf{k}} |0\rangle = 0$, the free vacuum state $|0\rangle$ has zero energy and zero photons. For a free electromagnetic field take $A^0 = 0$ since in Coulomb gauge Gauss' law leads to

$$\Phi(x) = A^0(x) = \frac{1}{4\pi} \int \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x' .$$

The polarization three-vectors are orthogonal to $\mathbf{k}$ and orthonormal to each other,

$$\epsilon(k, \lambda) \cdot \mathbf{k} = 0 \quad \text{and} \quad \epsilon(k, \lambda) \cdot \epsilon(k, \lambda') = \delta_{\lambda\lambda'} .$$

Since satisfying $\nabla \cdot \hat{\mathbf{A}} = 0$ we can write expand the field operator in terms of the plane wave states (normal modes) as

$$\hat{\mathbf{A}}(\mathbf{x}) = \int (d^3k/(2\pi)^3) \left(1/\sqrt{2\omega_{\mathbf{k}}}\right) \sum_{\lambda=1}^{2} \epsilon(k, \lambda) (\hat{a}^{\lambda}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^{\lambda\dagger}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}) .$$

Quantization of the electromagnetic field

(See Chapter 6, Secs 6.4.1 and 6.4.3)

Normal-ordered Hamiltonian: The result is that everything is consistent and we verify that the normal-ordered $\hat{H}$ is

$$\hat{H} = \int d^3x \, :\hat{\pi}^i \partial_0 \hat{A}_i - \mathscr{L}: \; = \; \int d^3x \, \frac{1}{2} :\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2: \; = \; \int [d^3k/(2\pi)^3] \omega_{\mathbf{k}} \sum_\lambda \hat{a}_{\mathbf{k}}^{\lambda\dagger} a_{\mathbf{k}}^{\lambda} \, .$$

In Coulomb gauge in the quantization inertial frame for a free photon $\hat{A}^0(x) = 0$ and so we can write

$$\hat{A}^\mu(x) = \int (d^3k/(2\pi)^3) \left(1/\sqrt{2\omega_{\mathbf{k}}}\right) \sum_{\lambda=1}^{2} \left(\epsilon^\mu(k,\lambda) \hat{a}_{\mathbf{k}}^{\lambda} e^{-ik\cdot x} + \epsilon^\mu(k,\lambda)^* \hat{a}_{\mathbf{k}}^{\lambda\dagger} e^{ik\cdot x}\right) \, ,$$

where we have defined

$$\epsilon^\mu(k,\lambda) \equiv \left(0, \epsilon(k,\lambda)\right) \quad \text{for} \quad \lambda = 1,2 \, .$$

We can write $\lambda = 1,2$ or $\lambda = \pm 1$ for the linear polarization and helicity bases respectively.

Functional integral for photons: This is discussed in detail in Secs. 6.4.3 and 6.4.4 but is too lengthy to try and summarize here. It is complicated by the need to restrict the physical space with the two constraints of Coulomb's law and Coulomb gauge. It is possible to generalize the argument to other gauges including especially covariant gauges that we will soon discuss. Covariant gauges maintain manifest Lorentz covariance during all stages of calculations and so are very convenient.

Arbitrary covariant gauge: It can be shown that we can quantize in an arbitary covariant gauge using for the spectral function with a source

$$F^j(t'',t') = \int \mathscr{D}A^\mu_{\text{periodic}} \exp[i\{S_\xi[A] - \int d^4x j_\mu A^\mu\}] \, , \quad \text{where}$$

$$S_\xi[A] \equiv \int d^4x \, \{ -(1/4) F_{\mu\nu} F^{\mu\nu} - (1/2\xi)(\partial_\mu A^\mu)^2 \} \, .$$

We refer to the arbitrary real ξ as the R_ξ *gauge parameter, where* $\xi = 1$ *is called Feynman gauge* and $\xi = 1$ *is called Landau gauge*.

Quantization of the electromagnetic field

(See Chapter 6, Secs 6.4.4)

Photon generating functional: We define the generating functional in the usual way as

$$Z[j] \equiv \lim_{T\to\infty(1-i\epsilon)} \frac{F^j(T,-T)}{F(T,-T)} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\{\hat{U}^j(T,-T)\}}{\mathrm{tr}\{\hat{U}(T,-T)\}} = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}A^\mu_{\text{periodic}}\, e^{i\{S_\xi[A]-\int_{-T}^{T} d^4x\, j_\mu A^\mu\}}}{\int \mathcal{D}A^\mu_{\text{periodic}}\, e^{iS_\xi[A]}}$$

For the free photon case we can write the action in a quadratic form

$$S_\xi[A] = \int d^4x\,\{-\tfrac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) - \tfrac{1}{2\xi}(\partial_\mu A^\mu)^2\}$$

$$= \int d^4x\, d^4y\, \delta^4(x-y)\tfrac{1}{2}\{-\partial^x_\mu A_\nu(x)\partial^\mu_y A^\nu(y) + \partial^\mu_x A_\nu(x)\partial^\nu_y A_\mu(y) - \tfrac{1}{\xi}\partial^\mu_x A_\mu(x)\partial^\nu_y A_\nu(y)\}$$

$$= \int d^4x\, d^4y\, \tfrac{1}{2}A_\mu(x)\big[\{g^{\mu\nu}\partial_x^2 - [1-\tfrac{1}{\xi}]\partial^\mu_x\partial^\nu_x\}\delta^4(x-y)\big]A_\nu(y) \equiv \int d^4x\, d^4y\, \tfrac{1}{2}A_\mu(x)K^{\mu\nu}(x,y)A_\nu(y)\,.$$

We can then perform the resulting Gaussian functional integrals to give

$$Z[j] = \exp\left\{-\tfrac{1}{2}\int d^4x\, d^4y\, j_\mu(x)D_F^{\mu\nu}(x-y)j_\nu(y)\right\}\,,$$ where

$$D_F^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+i\epsilon}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2+i\epsilon}\right)e^{-ik\cdot(x-y)} \equiv \int \frac{d^4k}{(2\pi)^4}D_F^{\mu\nu}(k)e^{-ik\cdot(x-y)}\,,$$

where $D_F^{\mu\nu}(x-y)$ is the *covariant Feynman propagator* for the photon. We see that $D_F^{\mu\nu}(x-y) = i(K^{-1})^{\mu\nu}(x,y)$ since

$$\{g^{\mu\nu}\partial_x^2 - [1-\tfrac{1}{\xi}]\partial^\mu_x\partial^\nu_x\}D_{F\nu\rho}(x-y) = i\delta^\mu{}_\rho\delta^4(x-y)\,,$$ which gives

$$\int d^4y\, K^{\mu\nu}(x,y)D_{F\nu\rho}(y-z) = i\delta^\mu{}_\rho\delta^4(x-z)\,.$$

Quantization of the electromagnetic field

(See Chapter 6, Secs 6.4.5)

$$\langle\Omega|T\hat{A}^{\mu_1}(x_1)\cdots\hat{A}^{\mu_k}(x_k)|\Omega\rangle = \frac{(i)^k\,\delta^k}{\delta j_{\mu_1}(x_1)\cdots\delta j_{\mu_k}(x_k)} Z[j]\Big|_{j=0} = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}A^\mu_{\text{periodic}}\, A^{\mu_1}(x_1)\cdots A^{\mu_k}(x_k)\, e^{iS_\xi[A]}}{\int \mathcal{D}A^\mu_{\text{periodic}}\, e^{iS_\xi[A]}}.$$

We then recover as expected

$$D_F^{\mu\nu}(x-y) = \langle\Omega|T\hat{A}^{\mu_1}(x_1)\cdots\hat{A}^{\mu_k}(x_k)|\Omega\rangle = (i)^2\frac{\delta^2}{\delta j_\mu(x)\delta j_\nu(y)} Z[j]\Big|_{j=0}.$$

As for the free scalar field case taking k derivatives vanishes unless k is even and gives the sum of every pairwise contraction of the μ_j, x_j pairs in the product of $k/2$ photon propagators as it should for bosons.

Covariant canonical quantization: This is discussed in detail in Sec. 6.4.5 and is referred to as the Gupta-Bleuler formalism. It uses Feynman gauge and also attempts to impose the covariant Lorenz gauge-fixing condition $\partial_\mu \hat{A}^\mu = 0$ to make the approach work. However we immediately have a conflict with the required canonical commutation relations for bosons. The Gupta-Bleuler approach imposes the weaker Lorenz gauge condition on the physical subspace that

$$\partial_\mu \hat{A}^{(+)}_\mu(x)|\Psi\rangle = 0 \quad \text{and} \quad \langle\Psi|\partial^\mu \hat{A}^{(-)}_\mu(x) \quad \text{for all} \quad |\Psi\rangle \in V_{\text{Phys}},$$

where $\hat{A}_\mu \equiv \hat{A}^{(+)}_\mu + \hat{A}^{(-)}_\mu$ with $\hat{A}^{(+)}_\mu$ and $\hat{A}^{(-)}_\mu$ corresponding to the annihilation (positive frequency) and creation (negative frequency) parts of $\hat{A}_\mu$ respectively. The procedure is successful and leads t

$$\langle 0|T\hat{A}^\mu(x)\hat{A}^\nu(y)|0\rangle = -g^{\mu\nu}D_F(x-y) = D_F^{\mu\nu}(x-y),$$

which is the Feynman gauge propagator.