# **Lecture 3:** Harmonic oscillator, normal modes, quantised free scalar field, Poincaré invariance, Fock space, functional integral, generating functional

# Quantization of normal modes

(See Chapter 6, Secs 6.1.1-6.1.2)

**Normal modes:** The derivation of normal modes for small oscillations around a static stable equilibrium for a classical systems was discussed in Chapter 2, Sec. 2.3. We denote the normal modes as $\vec{\zeta}(t)$, where $\vec{\zeta} = (\zeta_1, \ldots, \zeta_N)$ for a system with degrees $N$ of freedom. The effective Lagrangian and effective Hamiltonian are

$$L = \sum_{j=1}^{N} \frac{1}{2}\left(\dot{\zeta}_j^2 - \omega_j^2 \zeta_j^2\right), \qquad H = \sum_{j=1}^{N} \frac{1}{2}\left(\pi_j^2 + \omega_j^2 \zeta_j^2\right) = \sum_{j=1}^{N} \frac{1}{2}\left(\dot{\zeta}_j^2 + \omega_j^2 \zeta_j^2\right).$$

The normal mode conjugate momenta are $\pi_j \equiv \partial L/\partial \dot{\zeta}_j = \dot{\zeta}_j$ and the normal modes satisfy the harmonic oscillator equations of motion, $\ddot{\zeta}_j + \omega_j^2 \zeta_j^2 = 0$ . Any small oscillation around this stable equilibrium can be expressed as a linear superposition of the normal modes.

**Quantization of normal modes:** In Dirac's canonical quantization program the fundamental Poisson brackets become

$$[\hat{\zeta}_j(t), \hat{\zeta}_k(t)] = [\hat{\pi}_j(t), \hat{\pi}_k(t)] = 0 \quad \text{and} \quad [\hat{\zeta}_j(t), \hat{\pi}_k(t)] = i\hbar\delta_{jk} \; .$$

We have the normal mode Hamiltonian in quantum mechanics

$$\hat{H} = \sum_{j=1}^{N} \frac{1}{2}\left(\hat{\pi}_j^2 + \omega_j^2 \hat{\zeta}_j^2\right),$$

where $\hat{\zeta}_j$, $\hat{\pi}_j$ and $\omega_j$ are the normal mode Hermitian coordinate operator, the Hermitian conjugate momentum operator and the angular frequency of the $i^{\text{th}}$ normal mode respectively. An example of such quantized normal modes are the phonons of a crystalline solid, which arise from the quantized normal modes of the small oscillations of the ionic lattice of the crystal.

# Quantization of normal modes

(See Chapter 6, Secs 6.1.1-6.1.2)

Define the annihilation and creation operators as

$$\hat{a}_j \equiv \sqrt{\frac{\omega_j}{2\hbar}}\left(\hat{\zeta}_j + \frac{i}{\omega_j}\hat{\pi}_j\right) \quad \text{and} \quad \hat{a}_j^\dagger \equiv \sqrt{\frac{\omega_j}{2\hbar}}\left(\hat{\zeta}_j - \frac{i}{\omega_j}\hat{\pi}_j\right).$$

Then we have $[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0$ and $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and

$$\hat{H} = \sum_{j=1}^N \hbar\omega_j(\hat{a}_j\hat{a}_j^\dagger - \tfrac{1}{2}) = \sum_{j=1}^N \hbar\omega_j(\hat{a}_j^\dagger\hat{a}_j + \tfrac{1}{2}) \equiv \sum_{j=1}^N \hbar\omega_j(\hat{N}_j + \tfrac{1}{2}),$$

$$\hat{N}_j \equiv \hat{a}_j^\dagger\hat{a}_j, \quad \text{where} \quad \hat{N}_j = \hat{N}_j^\dagger \quad \text{and} \quad [\hat{N}_j, \hat{N}_k] = [\hat{H}, \hat{N}_j] = 0,$$

$$[\hat{N}_j, \hat{a}_k^\dagger] = \delta_{jk}\hat{a}_j^\dagger \quad \text{and} \quad [\hat{N}_j, \hat{a}_k] = -\delta_{jk}\hat{a}_j,$$

where $\hat{H}$ is the quantum Hamiltonian, $\hat{N}$ is the total number operator and $\hat{N}_j$ is the number operator for the $j^{\text{th}}$ normal mode. Define the number eigenstates (the occupancy number basis) as

$$|\vec{n}\,\rangle \equiv |n_1, n_2, \cdots, n_N\rangle \quad \text{where} \quad \hat{N}_j|\vec{n}\,\rangle = n_j|\vec{n}\,\rangle \quad \text{with} \quad n_j = 0,1,2,\ldots,$$

which satisfies

$$\hat{H}|\vec{n}\,\rangle = \sum_{j=1}^N \hbar\omega_j\left(\hat{N}_j + \tfrac{1}{2}\right)|\vec{n}\,\rangle = \sum_{j=1}^N \hbar\omega_j\left(n_j + \tfrac{1}{2}\right)|\vec{n}\,\rangle = E(\vec{n})|\vec{n}\,\rangle = E_0 + \sum_{j=1}^N n_j(\hbar\omega_j)$$

where $E_0 \equiv \sum_{j=1}^N \frac{1}{2}\hbar\omega_j$ is referred to as the *zero-point energy*.

It follows that

$$\hat{a}_j^\dagger|\cdots, n_j, \cdots\rangle = c_{j+}|\cdots, n_j + 1, \cdots\rangle, \qquad \hat{a}_j|\cdots, n_j, \cdots\rangle = c_{j-}|\cdots, n_j - 1, \cdots\rangle \ \text{ for } \ n_j \geq 1$$

with

$$c_{j+} = \sqrt{n_j + 1}, \qquad c_{j-} = \sqrt{n_j}.$$

So we see that

$\hat{a}_j^\dagger$ and $\hat{a}_j$ create and annihilate respectively a quasiparticle in the $j^{\text{th}}$ normal mode.

# Quantization of normal modes

(See Chapter 6, Secs 6.1.1-6.1.2)

So we see that we can form any occupation number basis state using

$$|\vec{n}\,\rangle = |n_1, n_2, \ldots, n_N\rangle = \prod_{j=1}^{N}\left((\hat{a}_j^\dagger)^{n_j}/\sqrt{n_j!}\right)|\vec{0}\,\rangle\,.$$

**Observation:** It seems reasonable to contemplate the possibility that what we think of as the classical vacuum is some stable equilibrium of a larger underlying theory, where what we think of as free particles are the quasiparticles of an effective free quantum theory built on that vacuum. Interactions are analogous to deviations from purely quadratic behavior.

**Use of natural units: F**rom this point on we will be using *natural units*, $\hbar = c = k = 1$, where $k$ is Boltzmann's constant. We may occasionally explicitly restore these constants when it is important to motivate a discussion. These quantities can always be restored from expressions in natural units using dimensional arguments.

**Free quantized scalar field:** A *scalar particle* is a boson (i.e., a particle that satisfies Bose-Einstein statistics) with zero spin. The quantized normal modes of the relativistic classical scalar field are the scalar particles of quantum field theory.

# Free scalar field

(See Chapter 6, Secs 6.2.1)

Recall the Lagrangian density of a free classical real scalar field,

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 = \tfrac{1}{2}\partial_0\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2$$

As we saw the corresponding Euler-Lagrange equation of motion is the Klein-Gordon equation given in \Eq{Eq:KG_eqn_class},

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = (\partial_0^2 - \nabla^2 + m^2)\phi(x) = 0\,.$$

Were it not for the $\nabla^2\phi(x)$ term these equations of motion would correspond to having one harmonic oscillator at each spatial point $\mathbf{x}$. So this system correspond to an infinite number of coupled harmonic oscillators. We also recall the Hamiltonian density from earlier,

$$\mathscr{H} = \tfrac{1}{2}\pi^2(x) + \tfrac{1}{2}\nabla^2\phi(x) + \tfrac{1}{2}m^2\phi(x) \quad \text{with} \quad \pi(x) \equiv \frac{\partial\mathscr{L}}{\partial(\partial_0\phi)}(x) = \partial^0\phi(x)$$

According to Dirac's canonical quantization procedure we should use the Correspondence Principle we should use the fundamental Poisson brackets for a classical scalar field theory,

$$\{\hat\phi(x),\hat\phi(y)\}_{x^0=y^0} = \{\hat\pi(x),\hat\pi(y\}_{x^0=y^0} = 0 \quad \text{and} \quad \{\hat\phi(x),\hat\pi(y)\}_{x^0=y^0} = i\delta^3(\mathbf{x}-\mathbf{y})\,.$$

to arrive at the equal-time canonical commutation relations (ETCR) for the quantum field theory,

$$[\hat\phi(x),\hat\phi(y)]_{x^0=y^0} = [\hat\pi(x),\hat\pi(y)]_{x^0=y^0} = 0 \quad \text{and} \quad [\hat\phi(x),\hat\pi(y)]_{x^0=y^0} = i\delta^3(\mathbf{x}-\mathbf{y})\,,$$

where these are the Heisenberg picture operators. *The operators will then obey the same equations of motion as their classical counterparts*. Recall that for a Hamiltonian with no explicit time dependence a Heisenberg picture operator, $\hat A(t)$, is related to its Schrödinger picture (denoted $s$) form $\hat A_s \equiv \hat A(t_0)$ by

$$\hat A(t) = e^{i\hat H(t-t_0)}\hat A_s\, e^{-i\hat H(t-t_0)}$$

where the reference time $t_0$ is the arbitrary time at which the two pictures coincide.

# Free scalar field

(See Chapter 6, Sec 6.2.1)

For a free scalar field the normal modes are the plane wave solutions, $\phi_{\mathbf{p}}(\mathbf{x},t)$, of the KGE,

$$\left[(\partial^2/\partial t^2) - \nabla^2 + m^2\right]\phi_{\mathbf{p}}(\mathbf{x},t) = \left[(\partial^2/\partial t^2) + \omega_{\mathbf{p}}^2\right]\phi_{\mathbf{p}}(\mathbf{x},t) = 0 \quad \text{for all } \mathbf{p} \in \mathbb{R}^3 ,$$

where $\omega_{\mathbf{p}} \equiv E_{\mathbf{p}} = \mathbf{p}^2 + m^2$. We have the correspondences to the discrete mechanics case

$$j \to \mathbf{p} , \quad \omega_j \to \omega_{\mathbf{p}} , \quad \zeta_j(t) \to \phi_{\mathbf{p}}(\mathbf{x},t) ,$$

which leads to

$$\hat{a}_j \;\to\; \hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} \qquad \text{and} \qquad \hat{a}_j^\dagger \;\to\; \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} ,$$

$$\hat{\zeta}_j = (1/\sqrt{2\omega_j})(\hat{a}_j + \hat{a}_j^\dagger) \;\to\; \hat{\phi}_{\mathbf{p}}(\mathbf{x}) = (1/\sqrt{2\omega_{\mathbf{p}}})\left(\hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) ,$$

$$\hat{\pi}_j = -i\sqrt{\omega_j/2}(\hat{a}_j - \hat{a}_j^\dagger) \;\to\; \hat{\pi}_{\mathbf{p}}(\mathbf{x}) = -i\sqrt{\omega_{\mathbf{p}}/2}\left(\hat{a}_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right) .$$

From Dirac's canonical quantization procedure we arrive at (in the Schrödinger representation)

$$[\hat{\phi}_s(\mathbf{x}), \hat{\phi}_s(\mathbf{y})] = \left[\hat{\pi}_s(\mathbf{x}), \hat{\pi}_s(\mathbf{y})\right] = 0 \quad \text{and} \quad [\hat{\phi}_s(\mathbf{x}), \hat{\pi}_s(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}) .$$

This leads to the creation and annihilation operator commutation relations

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}] = [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}'}^\dagger] = 0 \quad \text{and} \quad [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$$

in analogy with the mechanics case. The *number density operator* for the normal mode labeled by $\mathbf{p}$ is

$$\hat{N}_{\mathbf{p}} \equiv \frac{1}{(2\pi)^3}\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} ,$$

$$\Rightarrow \quad [\hat{N}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] = \hat{a}_{\mathbf{p}}^\dagger \delta^3(\mathbf{p}-\mathbf{p}') , \quad [\hat{N}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}] = -\hat{a}_{\mathbf{p}}\delta^3(\mathbf{p}-\mathbf{p}') , \quad [\hat{N}_{\mathbf{p}}, \hat{N}_{\mathbf{p}'}] = 0 .$$

The *number operator,* $\hat{N}$, includes all normal modes,

$$\hat{N} \equiv \int d^3p \, \hat{N}_{\mathbf{p}} = \int (d^3p/(2\pi)^3)\, \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} .$$

# Free scalar field

(See Chapter 6, Sec 6.2.1)

The Hamiltonian is given by

$$\hat{H} = \int d^3x\, \mathscr{H} = \int d^3x \left[\frac{1}{2}\hat{\pi}_s^2 + \frac{1}{2}(\nabla\hat{\phi}_s)^2 + \frac{1}{2}m^2\hat{\phi}_s^2\right]$$

$$= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \frac{1}{2}\left\{ -\frac{\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}{2}(\hat{a}_\mathbf{p} - \hat{a}^\dagger_{-\mathbf{p}})(\hat{a}_{\mathbf{p}'} - \hat{a}^\dagger_{-\mathbf{p}'})\right.$$

$$\left. + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{2\sqrt{\omega_\mathbf{p}\omega_{\mathbf{p}'}}}(\hat{a}_\mathbf{p} + \hat{a}^\dagger_{-\mathbf{p}})(\hat{a}_{\mathbf{p}'} + \hat{a}^\dagger_{-\mathbf{p}'})\right\} = \int \frac{d^3p\, d^3p'}{(2\pi)^3}\delta^3(\mathbf{p}+\mathbf{p}')\frac{1}{2}\left\{\cdots\right\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_\mathbf{p}}{2}(\hat{a}^\dagger_{-\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{p}) = \int \frac{d^3p}{(2\pi)^3}\,\omega_\mathbf{p}(\hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{p} + \frac{1}{2}[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{p}]),$$

where the very last term is the infinite zero-point energy,

$$\langle 0|\hat{H}|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{\omega_\mathbf{p}}{2}\langle 0|(\hat{a}^\dagger_{-\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{p})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{\omega_\mathbf{p}}{2}[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{p}] = \int d^3p\,\frac{\omega_\mathbf{p}}{2}\delta^3(\mathbf{0}).$$

Subtracting a constant from the Hamiltonian has no physical consequence and so we can always do this. It is then convenient to define *normal ordering*, where all of the creation operators in a product are moved to the left of all of the annihilation operators. The normal ordered form of an operator $\hat{A}$ is denoted $:\hat{A}:$ and we then have for example,

$$:\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}:=:\hat{a}_{\mathbf{p}_2}\hat{a}^\dagger_{\mathbf{p}_1}:=\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2} \quad \text{and} \quad :\hat{a}_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}\hat{a}^\dagger_{\mathbf{p}_3}\hat{a}_{\mathbf{p}_4}\hat{a}_{\mathbf{p}_5}\hat{a}^\dagger_{\mathbf{p}_6}:=\hat{a}^\dagger_{\mathbf{p}_2}\hat{a}^\dagger_{\mathbf{p}_3}\hat{a}^\dagger_{\mathbf{p}_6}\hat{a}_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_4}\hat{a}_{\mathbf{p}_5}.$$

Normal ordering is not a linear operation and we *define* for example

$$:\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}\hat{a}_{\mathbf{p}_3} + \hat{a}_{\mathbf{p}_4}\hat{a}_{\mathbf{p}_5}\hat{a}_{\mathbf{p}_6}: \;\equiv\; :\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}\hat{a}_{\mathbf{p}_3}: + :\hat{a}_{\mathbf{p}_4}\hat{a}_{\mathbf{p}_5}\hat{a}_{\mathbf{p}_6}:$$

# Free scalar field

(See Chapter 6, Sec 6.2.1)

Since $\hat{a}_{\mathbf{p}}|0\rangle = 0$ and $\langle 0|\hat{a}^\dagger_{\mathbf{p}} = 0$ then any normal-ordered operator containing at least one $\hat{a}_{\mathbf{p}}$ or $\hat{a}^\dagger_{\mathbf{p}}$ will have a vanishing vacuum expectation value, $\langle 0|\ {:}f(\hat{a},\hat{a}^\dagger){:}\ |0\rangle = 0$.

So we simply *redefine* the Hamiltonian to be its normal-ordered form,

$$\hat{H} \equiv \int d^3x\ {:}\left[\tfrac{1}{2}\hat{\pi}_s^2 + \tfrac{1}{2}(\nabla\hat{\phi}_s)^2 + \tfrac{1}{2}m^2\hat{\phi}_s^2\right]{:} = \int (d^3p/(2\pi)^3)\,(\omega_{\mathbf{p}}/2)\ {:}\hat{a}^\dagger_{-\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}{:}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} = \int d^3p\ \omega_{\mathbf{p}}\hat{N}_{\mathbf{p}} = \int d^3p\ E_{\mathbf{p}}\hat{N}_{\mathbf{p}}\,,$$

which satisfies $\langle 0|\hat{H}|0\rangle = 0$ . Similarly the three-momentum operator to be its normal-ordered form,

$$\hat{\mathbf{P}} \equiv -\int d^3x\ {:}\hat{\pi}\nabla\hat{\phi}{:} = \int (d^3p/(2\pi)^3)\,\mathbf{p}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}) = \int d^3p\ \mathbf{p}\,\hat{N}_{\mathbf{p}}\,,$$ which satisfies $\langle 0|\hat{\mathbf{P}}|0\rangle = 0$ .

In summary we can write $\hat{P}^\mu \equiv (\hat{H},\hat{\mathbf{P}}) = \int d^3p\ p^\mu\hat{N}_{\mathbf{p}}$ and $\langle 0|\hat{P}^\mu|0\rangle = 0$ .

Using the earlier commutation relations for $\hat{N}_{\mathbf{p}}$, $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$ it is then relatively simple to show that

$$[\hat{N},\hat{a}^\dagger_{\mathbf{p}}] = \hat{a}^\dagger_{\mathbf{p}}\,,\quad [\hat{N},\hat{a}_{\mathbf{p}}] = -\,\hat{a}_{\mathbf{p}}\,,\quad [\hat{N},\hat{N}_{\mathbf{p}}] = 0\,,\quad [\hat{H},\hat{N}_{\mathbf{p}}] = [\hat{H},\hat{N}] = 0\,,$$

$$[\hat{H},\hat{a}^\dagger_{\mathbf{p}}] = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} \equiv E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\,,\quad [\hat{H},\hat{a}_{\mathbf{p}}] = -\,\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}} \equiv -\,E_{\mathbf{p}}\hat{a}_{\mathbf{p}}\,,$$

$$[\hat{\mathbf{P}},\hat{a}^\dagger_{\mathbf{p}}] = \mathbf{p}\hat{a}^\dagger_{\mathbf{p}}\,,\quad [\hat{\mathbf{P}},\hat{a}_{\mathbf{p}}] = -\,\mathbf{p}\hat{a}_{\mathbf{p}}\,,$$

$$[\hat{H},\hat{P}^\mu] = 0\,,\quad [\hat{P}^\mu,\hat{a}^\dagger_{\mathbf{p}}] = p^\mu\Big|_{p^0=E_{\mathbf{p}}} a^\dagger_{\mathbf{p}} \quad\text{and}\quad [\hat{P}^\mu,\hat{a}_{\mathbf{p}}] = -p^\mu\Big|_{p^0=E_{\mathbf{p}}} a_{\mathbf{p}}$$

Note that

$$\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}|0\rangle = [\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{q}}]|0\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})|0\rangle\,,\qquad \langle 0|\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}|0\rangle = [\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\,.$$

# Free scalar field

(See Chapter 6, Sec 6.2.3)

This result generalizes to

$$\langle 0|\hat{a}_{\mathbf{p}_n}\cdots\hat{a}_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{q}_1}\cdots\hat{a}^\dagger_{\mathbf{q}_n}|0\rangle = (2\pi)^{3n}\left[\delta^3(\mathbf{p}_1-\mathbf{q}_1)\cdots\delta^3(\mathbf{p}_n-\mathbf{q}_n) + \text{ all permutations of } (\mathbf{q}_1\cdots\mathbf{q}_n)\right] .$$

**Normalization of states:** The single boson state with three-momentum $\mathbf{p}$ and energy $E_\mathbf{p} = (\mathbf{p}^2 + m^2)^{1/2}$ is defined to have the normalization

$$|\mathbf{p}\rangle \equiv \sqrt{2E_\mathbf{p}}\,\hat{a}^\dagger_\mathbf{p}|0\rangle ,$$ which leads to the result that $$\langle \mathbf{p}|\mathbf{q}\rangle = 2E_\mathbf{p}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) .$$

**Summary:** Each application of $\hat{a}^\dagger_\mathbf{p}$ creates an on-shell particle with energy $E_\mathbf{p} = (|\mathbf{p}|^2 + m^2)^{1/2}$ and momentum $\mathbf{p}$ and so $\hat{a}^\dagger_\mathbf{p}$ is a *creation operator*. Conversely, each application of $\hat{a}_\mathbf{p}$ removes an on-shell particle with energy if there is one and so $\hat{a}_\mathbf{p}$ is an *annihilation (or destruction)} operator*. If there is no corresponding particle to annihilate, then $\hat{a}_\mathbf{p}$ acting on the state gives zero.

**Fock space:** The space of all free particle states is referred to as *Fock space,* which is the Hilbert space for the free field. A symmetrized state is an immediate consequence of applying $n$normalized one-boson creation operators to give

$$|\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_n\rangle \equiv 2^{n/2}\sqrt{E_{\mathbf{p}_1}E_{\mathbf{p}_2}\cdots E_{\mathbf{p}_n}}\,\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}^\dagger_{\mathbf{p}_2}\cdots\hat{a}^\dagger_{\mathbf{p}_n}|0\rangle .$$

The state obeys Bose-Einstein statistics (symmetric under particle exchange) since all of the creation operators $\hat{a}^\dagger_\mathbf{p}$ commute with each other. The set of all such states for different momenta $\mathbf{p}_j$ and particle (boson) number $n$ are a basis for Fock space. The set of all $\mathbf{p}_j$ for some fixed $n$ form a basis for the $n$ -boson subspace of the Fock space.

# Free scalar field

(See Chapter 6, Sec 6.2.3)

The combination $d^3p/E_{\mathbf{p}}$ is Lorentz invariant, $d^3p'/E_{\mathbf{p}'} = d^3p/E_{\mathbf{p}}$, [see proof of Eq. (6.2.50)] and we can show that for any Lorentz invariant function of four-momentum , $g(p)$, we have the result that

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\, g(p)\,|_{p^0=\pm E_{\mathbf{p}}} = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2-m^2)g(p)\Big|_{p^0\gtrless 0}.$$

It also follows that the field operator acting on the vacuum creates a one-particle state at $x$,

$$\hat{\phi}(x)\,|\,0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{ip\cdot x}\hat{a}^\dagger_{\mathbf{p}}\,|\,0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\, e^{ip\cdot x}\Big|_{p^0=E_{\mathbf{p}}}\,|\,\mathbf{p}\rangle\,.$$

**Lagrangian and equations of motion:** Dirac's canonical quantization program leads to the operator Lagrangian and equations of motion

$$\hat{H} \equiv \int d^3x\, \hat{\mathscr{H}} \equiv \left(\int d^3x\, \hat{\pi}(x)\dot{\hat{\phi}}(x)\right) - \hat{L} = \int d^3x\, \left(\hat{\pi}(x)\dot{\hat{\phi}}(x) - \hat{\mathscr{L}}\right)$$

$$\hat{\pi}(x) \equiv \frac{\delta \hat{L}}{\delta(\partial_0\hat{\phi})(x)} = \frac{\partial \hat{\mathscr{L}}}{\partial(\partial_0\hat{\phi})(x)} \qquad \text{and} \qquad \partial_\mu\left(\frac{\partial\hat{\mathscr{L}}}{\partial(\partial_\mu\hat{\phi})(x)}\right) - \frac{\partial\hat{\mathscr{L}}}{\partial\hat{\phi}(x)} = 0\,.$$

# Free scalar field

(See Chapter 6, Sec 6.2.4)

We also recover the four-momentum and angular momentum operators as before which obey the Poincaré Lie algebra in operator form in terms of commutators,

$$\hat{T}^{\mu\nu}(x) = \hat{\vec{\pi}}^{\mu} \cdot \partial^{\nu}\hat{\vec{\phi}} - g^{\mu\nu}\hat{\mathscr{L}}\,,$$

$$\hat{P}^{\mu} = \int d^3x\, \hat{T}^{0\mu}(x) = (\hat{H}, \hat{\mathbf{P}}) \;\text{ with }\; \hat{H} = \int d^3x\, [\hat{\vec{\pi}} \cdot \partial^0 \hat{\vec{\phi}} - \hat{\mathscr{L}}]\,,\;\; \hat{P}^i = \int d^3x\, \hat{\vec{\pi}} \cdot \partial^i \hat{\vec{\phi}}\,,$$

$$\hat{M}^{\mu\nu} = \int d^3x\, [x^{\mu}\hat{T}^{0\nu}(x) - x^{\nu}\hat{T}^{0\mu}(x)]\,,$$

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = i(g^{\nu\rho}\hat{M}^{\mu\sigma} - g^{\mu\rho}\hat{M}^{\nu\sigma} - g^{\nu\sigma}\hat{M}^{\mu\rho} + g^{\mu\sigma}\hat{M}^{\nu\rho})\,,$$

$$[\hat{P}^{\mu}, \hat{M}^{\rho\sigma}] = i(g^{\mu\rho}\hat{P}^{\sigma} - g^{\mu\sigma}\hat{P}^{\rho})\,, \qquad [\hat{P}^{\mu}, \hat{P}^{\nu}] = 0\,.$$

Where we have a unitary representation of the translation operator and the restricted Lorentz transformations of $SO^{+}(1,3)$ given respectively by

$$\hat{U}(a) = e^{i\hat{P}\cdot a} \quad \text{and} \quad \hat{U}(\Lambda) = e^{-(i/2)\omega_{\mu\nu}\hat{M}^{\mu\nu}}\,.$$

**Causality and spacelike separations:** For space like separations we can show

$$i\Delta(x-y) \equiv [\hat{\phi}(x), \hat{\phi}(y)] = 0 \quad \text{for all} \quad (x-y)^2 < 0$$ [see proof of Eq. (6.2.144)] .

So the field operator commutes with itself whenever the spacetime points have a spacelike separation. This is the condition of causality in that the measurement of the field at point $x$ cannot affect the measurement of the field at point $y$for a spacelike separation.

**Time-ordering operator:** Recall the time-ordering operator introduced in Lecture 1,

$$T\hat{A}(x)\hat{B}(y) \equiv \theta(x^0 - y^0)\hat{A}(x)\hat{B}(y) + \theta(y^0 - x^0)\hat{B}(y)\hat{A}(x)\,.$$

# Free scalar field

(See Chapter 6, Sec 6.2.6)

**Feynman propagator:** The Feynman propagator for the scalar field is defined as the vacuum expectation value (vev) of the time-ordered product of two field operators,

$$D_F(x-y) \equiv \langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)} \equiv \int \frac{d^4p}{(2\pi)^4}D_F(p)e^{-ip.x},$$

where the result on the right-hand side is obtained after a little work [see proof of Eq. (6.2.153)] and where $D_F(p) \equiv i/(p^2-m^2+i\epsilon)$ is the momentum-space Feynman propagator. The $i\epsilon$ in the denominator results from analytically continuing back from Euclidean space and acts as a damping term in the functional integral. It has the same origin as it did in quantum mechanics.

We can decompose the positive and negative energy parts of the field as $\hat{\phi}(x) = \hat{\phi}^+(x) + \hat{\phi}^-(x)$, where

$$\hat{\phi}^+(x) \equiv \int \frac{d^3p}{(2\pi)^3}\left(1/\sqrt{2E_p}\right)\hat{a}_{\mathbf{p}}e^{-ip\cdot x} \quad \text{and} \quad \hat{\phi}^-(x) \equiv \int \frac{d^3p}{(2\pi)^3}\left(1/\sqrt{2E_p}\right)\hat{a}^\dagger_{\mathbf{p}}e^{ip\cdot x}.$$

We see that $\hat{\phi}^+(x)|0\rangle = 0\,,\ \langle 0|\hat{\phi}^-(x) = 0 \ \Rightarrow\ \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \langle 0|\hat{\phi}^+(x)\hat{\phi}^-(y)|0\rangle$.

**Contractions:** We define the *contraction* of two field operators as

$$\overbracket{\hat{\phi}(x)\hat{\phi}}(y) \equiv \begin{cases} [\hat{\phi}^+(x),\hat{\phi}^-(y)] & \text{for } x^0 > y^0 \\ [\hat{\phi}^+(y),\hat{\phi}^-(x)] & \text{for } x^0 < y^0 \end{cases}.$$

It then follows that

$$T\hat{\phi}(x)\hat{\phi}(y) = :\hat{\phi}(x)\hat{\phi}(y): + D_F(x-y) = :\hat{\phi}(x)\hat{\phi}(y) + D_F(x-y):$$

$$= :\hat{\phi}(x)\hat{\phi}(y) + \overbracket{\hat{\phi}(x)\hat{\phi}}(y):,$$

# Free scalar field

(See Chapter 6, Sec 6.2.8)

**Wick's theorem:** The generalization of this result is referred to as *Wick's Theorem* and has the form

$$T\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) = \,:\{\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) + \text{all contractions}\}: \quad .$$

For a proof see the proof of Eq. (6.2.208).

Consider an example of Wick's theorem

$$\begin{aligned}
T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) = \,:\Big(&\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4) + \wick{\c1{\hat{\phi}}(x_1)\c1{\hat{\phi}}(x_2)}\hat{\phi}(x_3)\hat{\phi}(x_4) \\
&+ \wick{\c1{\hat{\phi}}(x_1)\hat{\phi}(x_2)\c1{\hat{\phi}}(x_3)}\hat{\phi}(x_4) + \wick{\c1{\hat{\phi}}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\c1{\hat{\phi}}(x_4)} + \hat{\phi}(x_1)\wick{\c1{\hat{\phi}}(x_2)\c1{\hat{\phi}}(x_3)}\hat{\phi}(x_4) \\
&+ \hat{\phi}(x_1)\wick{\c1{\hat{\phi}}(x_2)\hat{\phi}(x_3)\c1{\hat{\phi}}(x_4)} + \hat{\phi}(x_1)\hat{\phi}(x_2)\wick{\c1{\hat{\phi}}(x_3)\c1{\hat{\phi}}(x_4)} + \wick{\c1{\hat{\phi}}(x_1)\c1{\hat{\phi}}(x_2)\c2{\hat{\phi}}(x_3)\c2{\hat{\phi}}(x_4)} \\
&+ \wick{\c1{\hat{\phi}}(x_1)\c2{\hat{\phi}}(x_2)\c1{\hat{\phi}}(x_3)\c2{\hat{\phi}}(x_4)} + \wick{\c1{\hat{\phi}}(x_1)\c2{\hat{\phi}}(x_2)\c2{\hat{\phi}}(x_3)\c1{\hat{\phi}}(x_4)}\Big): \; . \qquad (6.2.209)
\end{aligned}$$

Taking the vacuum expectation value of this leaves only the fully contracted terms, since the vacuum expectation value of any normal-ordered product of operators vanishes, $\langle 0|:\hat{\phi}(x_1)\cdots\hat{\phi}(x_j):|0\rangle = 0$. So

$$\begin{aligned}
\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4) \\
&+ D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3) \; .
\end{aligned}$$

**Consequence of Wick's Theorem:** The general form of this result is that the vacuum expectation value of the time-ordered product of operators will vanish if $n$ is odd and if $n$ is even the result is

$$\begin{aligned}
\langle 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)|0\rangle &= D_F(x_1-x_2)D_F(x_3-x_4)\cdots D_F(x_{n-1}-x_n) \\
&+ \text{all pairwise combinations of } (x_1, x_2, \cdots, x_n)
\end{aligned}$$

# Free scalar field

(See Chapter 6, Sections 6.2.9 and 6.2.11)

**Functional integral formulation:** The extension of the path/functional in quantum mechanics to scalar quantum field theory is relatively straightforward and we arrive at an analogous result,

$$\langle\phi'',t''|\phi',t'\rangle = \langle\phi''|\hat{G}(t''-t')|\phi'\rangle = \langle\phi''|e^{-i\hat{H}t/\hbar}|\phi'\rangle = \int\mathscr{D}\phi\, e^{iS[\phi]/\hbar},$$

which leads to the result for the spectral function

$$F(t''-t') \equiv \operatorname{tr} e^{-i\hat{H}(t''-t')/\hbar} = \int\mathscr{D}^{\rm sp}\phi\,\langle\phi|e^{-i\hat{H}(t''-t')/\hbar}|\phi\rangle = \int\mathscr{D}^{\rm sp}\phi\,\langle\phi,t''|\phi,t'\rangle$$

$$= \int\mathscr{D}\pi\,\mathscr{D}\phi_{\rm (periodic)}\,\exp\{(i/\hbar)\int_{t'}^{t''}d^4x\,[\pi\dot{\phi}-\mathscr{H}(\phi,\pi)]\}$$

$$= \int\mathscr{D}\phi_{\rm (periodic)}\,\exp\{(i/\hbar)\int_{t'}^{t''}d^4x\,\mathscr{L}(\phi,\partial_\mu\phi\} = \int\mathscr{D}\phi_{\rm (periodic)}\,\exp\{(i/\hbar)S([\phi],t'',t')\}.$$

The integral is over functions $\phi(x)$ periodic in space (or vanishing at spatial infinity) and periodic in time.

**Spectral function for a scalar field with a source:** Including a source term $j(x)\phi(x)$ for the scalar field again leads to a result analogous to that for quantum mechanics for the spectral function with a source $j$,

$$F^j(t'',t') = \operatorname{tr}\{\hat{U}^j(t'',t')\} = \operatorname{tr}\left\{Te^{-i\int_{t'}^{t''}d^4x[\hat{\mathscr{H}}-j(x)\hat{\phi}(x)]}\right\} = \int\mathscr{D}\phi_{\rm (periodic)}\,e^{iS[\phi,j]},$$

where we have defined

$$S[\phi,j] \equiv S[\phi]+\int_{t'}^{t''}d^4x\,j(x)\phi(x) = \int_{t'}^{t''}d^4x\left[\mathscr{L}+j(x)\phi(x)\right] \quad \text{and} \quad \mathscr{L} = \tfrac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2] - U(\phi).$$

We will now denote the general vacuum state as $|\Omega\rangle$ and use that notation in interacting theories and we will define it to be normalized such that $\langle\Omega|\Omega\rangle = 1$. In a free theory we have $|\Omega\rangle = |0\rangle$.

# Free scalar field generating functional

(See Chapter 6, Sections 6.2.9 and 6.2.11)

**Generating functional:** In analogy with quantum mechanics we define the generating functional $Z[j]$ as

$$Z[j] \equiv \lim_{T\to\infty(1-i\epsilon)} \frac{F^j(T,-T)}{F(T,-T)} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\{\hat{U}^j(T,-T)\}}{\mathrm{tr}\{\hat{U}(T,-T)\}} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\left[Te^{-i\int_{-T}^{T}d^4x\,[\hat{\mathscr{H}}-j\hat{\phi}]}\right]}{\mathrm{tr}\left[Te^{-i\int_{-T}^{T}d^4x\,\hat{\mathscr{H}}}\right]}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle\Omega|Te^{i\int_{-T}^{T}d^4x\,j(x)\hat{\phi}_I(x)}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathscr{D}\phi_{(\text{periodic})}\,e^{i\{S[\phi]+\int_{-T}^{T}d^4x\,j\phi\}}}{\int\mathscr{D}\phi_{(\text{periodic})}\,e^{iS[\phi]}}.$$

Similarly it can then be shown by generalizing the quantum mechanics arguments that the ground state expectation value of time-ordered products of Heisenberg picture scalar operators are given by

$$\langle\Omega|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_k)|\Omega\rangle = \left.\frac{(-i)^k\delta^k}{\delta j(x_1)\cdots\delta j(x_k)}Z[j]\right|_{j=0} = \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathscr{D}\phi_{(\text{periodic})}\,\phi(x_1)\cdots\phi(x_k)\,e^{iS[\phi]}}{\int\mathscr{D}\phi_{(\text{periodic})}\,e^{iS[\phi]}}.$$

One commonly finds that the limit $T\to\infty(1-i\epsilon)$ and ``periodic" are not explicitlyindicated but are to be understood and that the denominator is sometimes not included in the definition of $Z[j]$ , e.g., in Peskin and Schroeder. It will later be useful to also define the time-ordered product *before* taking $j(x)=0$ ,

$$\langle\Omega|T\hat{\phi}(x_1)\cdots\hat{\phi}(x_k)|\Omega\rangle_j \equiv \frac{(-i)^k\delta^k}{\delta j(x_1)\cdots\delta j(x_k)}Z[j]$$

and so, for example, $Z[j]\equiv\langle\Omega|\Omega\rangle_j$.

# Free scalar field generating functional

(See Chapter 6, Sections 6.2.9 and 6.2.11)

**Summary for free scalar field:** We have for the Minkowski space action with source $j(x)$ that

$$S[\phi, j] = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - (m^2 - i\epsilon)\phi^2 + j\phi \right\}$$

$$= \int d^4x\, d^4y \left[ \frac{1}{2}\phi(x) \left\{ \left( -\partial^x_\mu\partial^{x\mu} - m^2 + i\epsilon \right) \delta^4(x-y) \right\} \phi(y) \right] + \int d^4x\, j\phi$$

For a free scalar field we see above that the action is a gaussian and after carrying out the gaussian integration over fields we obtain

$$Z[j] = \exp\left\{ -\frac{1}{2} \int d^4x\, d^4y\, j(x)\, D_F(x-y) j(y) \right\} .$$

When we evaluate the two-point function we then see that we recover as we should that

$$\langle 0 | T\hat{\phi}(x)\hat{\phi}(y) | 0 \rangle = D_F(x-y) .$$

More generally we recover the result of Wick's theorem from the free scalar field generating functional $Z[j]$, since

$$\langle 0 | T\hat{\phi}(x_1)\ldots\hat{\phi}(x_k) | 0 \rangle = (-i)^k \frac{\delta^k}{\delta j(t_1)\cdots\delta j(t_k)} Z[j] \Bigg|_{j=0}$$

$$= \delta_{[k=\mathrm{even}]} \left\{ D_F(x_1 - x_2) D_F(x_3 - x_4) \cdots D_F(x_{k-1} - x_k) \right.$$

$$\left. + \text{ all pairwise combinations of } (x_1, x_2, \cdots, x_k) \right\} .$$

# Free charged scalar field
(See Chapter 6, Sections 6.2.7)

**Free charged scalar field:** We now wish to consider the charged scalar field as a quantum field theory. We can write the normal-ordered Lagangian density operator in terms of Heisenberg picture field operators as

$$\hat{\mathscr{L}} = \frac{1}{2} :\left[(\partial_\mu\hat{\phi}_1)^2 + (\partial_\mu\hat{\phi}_2)^2 - m^2\hat{\phi}_1^2 - m^2\hat{\phi}_2^2\right]: \; = \; :\partial_\mu\hat{\phi}^\dagger\partial^\mu\hat{\phi} - m^2\hat{\phi}^\dagger\hat{\phi}: \; ,$$

where we have defined $\hat{\phi}(x) \equiv (1/\sqrt{2})[\hat{\phi}_1(x) + i\hat{\phi}_2(x)]$ , $\hat{\phi}^\dagger(x) \equiv (1/\sqrt{2})[\hat{\phi}_1(x) - i\hat{\phi}_2(x)]$ ,and where we have the usual cancel commutation relations

$$[\hat{\phi}_i(t,\mathbf{x}), \hat{\phi}_j(t,\mathbf{y})] = [\hat{\pi}_i(t,\mathbf{x}), \hat{\pi}_j(t,\mathbf{y})] = 0 \,, \quad [\hat{\phi}_i(t,\mathbf{x}), \hat{\pi}_j(t,\mathbf{y})] = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \,.$$

These can equally be written in terms of the $\hat{\phi}, \hat{\phi}^\dagger, \hat{\pi}, \hat{\pi}^\dagger$. The *only non-vanishing commutators* are

$$[\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{x}')] = [\hat{\phi}^\dagger(t,\mathbf{x}), \hat{\pi}^\dagger(t,\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}') \,.$$

The operators must obey the same equations as their classical counterparts so we have

$$\hat{\pi}(x) = \frac{\partial\hat{\mathscr{L}}}{\partial(\partial_0\hat{\phi}(x))} = \partial^0\hat{\phi}^\dagger(x) \,, \qquad \hat{\pi}^\dagger(x) = \frac{\partial\hat{\mathscr{L}}}{\partial(\partial_0\hat{\phi}^\dagger(x))} = \partial^0\hat{\phi}(x) \,.$$

The normal-ordered Hamiltonian density is obtained in the usual way

$$\hat{\mathscr{H}} = \; :\dot{\hat{\phi}}\hat{\pi} + \hat{\pi}^\dagger\dot{\hat{\phi}}^\dagger - \hat{\mathscr{L}}: \; = \; :\hat{\pi}^\dagger\hat{\pi} + (\nabla\hat{\phi}^\dagger)\cdot(\nabla\hat{\phi}) + m^2\hat{\phi}^\dagger\hat{\phi}:$$

From Noether's Theorem we recover in the usual way the normal-ordered Hamiltonian and three-momentum operators, which are given by

$$\hat{H} = \int d^3x \;:\hat{\pi}^\dagger\hat{\pi} + (\nabla\hat{\phi}^\dagger)\cdot(\nabla\hat{\phi}) + m^2\hat{\phi}^\dagger\hat{\phi}: \; ,$$

$$\hat{P} = -\int d^3x \;:\hat{\pi}\nabla\hat{\phi} + \hat{\pi}^\dagger\nabla\hat{\phi}^\dagger: \; .$$

# Free charged scalar field

(See Chapter 6, Sections 6.2.7)

We can defined new annihilation and creation operators for particles associated with $\phi$ (denoted $\hat{f}_{\mathbf{p}}, \hat{f}^\dagger_{\mathbf{p}}$) and antiparticles associated with $\phi^\dagger$ (denoted $\hat{g}_{\mathbf{p}}, \hat{g}^\dagger_{\mathbf{p}}$),

$$\hat{f}_{\mathbf{p}} \equiv (1/\sqrt{2})[\hat{a}_{1\mathbf{p}} + i\hat{a}_{2\mathbf{p}}]\,, \qquad \hat{f}^\dagger_{\mathbf{p}} \equiv (1/\sqrt{2})[\hat{a}^\dagger_{1\mathbf{p}} - i\hat{a}^\dagger_{2\mathbf{p}}]\,,$$

$$\hat{g}_{\mathbf{p}} \equiv (1/\sqrt{2})[\hat{a}_{1\mathbf{p}} - i\hat{a}_{2\mathbf{p}}]\,, \qquad \hat{g}^\dagger_{\mathbf{p}} \equiv (1/\sqrt{2})[\hat{a}^\dagger_{1\mathbf{p}} + i\hat{a}^\dagger_{2\mathbf{p}}]\,.$$

It is straightforward to verify that the commutation relations between the new annihilation and creation operators are

$$[\hat{f}_{\mathbf{p}}, \hat{f}_{\mathbf{p}'}] = [\hat{f}^\dagger_{\mathbf{p}}, \hat{f}^\dagger_{\mathbf{p}'}] = [\hat{g}_{\mathbf{p}}, \hat{g}_{\mathbf{p}'}] = [\hat{g}^\dagger_{\mathbf{p}}, \hat{g}^\dagger_{\mathbf{p}'}] = [\hat{f}_{\mathbf{p}}, \hat{g}_{\mathbf{p}'}] = [\hat{f}_{\mathbf{p}}, \hat{g}^\dagger_{\mathbf{p}'}] = [\hat{f}^\dagger_{\mathbf{p}}, \hat{g}_{\mathbf{p}'}] = [\hat{f}^\dagger_{\mathbf{p}}, \hat{g}^\dagger_{\mathbf{p}'}] = 0,$$

$$[\hat{f}_{\mathbf{p}}, \hat{f}^\dagger_{\mathbf{p}'}] = [\hat{g}_{\mathbf{p}}, \hat{g}^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')\,.$$

**Noether current (from phase invariance):** From the classical Noether current we know that we will have the conserved normal-ordered Noether current at the operator level,

$$\hat{j}^\mu(x) = i : \left[\hat{\phi}^\dagger(\partial^\mu\hat{\phi}) - (\partial^\mu\hat{\phi}^\dagger)\hat{\phi}\right] : \quad \text{with} \quad \partial_\mu \hat{j}^\mu(x) = 0\,,$$

where we have chosen to change the sign of the current remembering that we are always free to redefine a conserved current by multiplying it by a constant.

**Conserved charge:** The corresponding conserved charge is then

$$\hat{Q} = \int d^3x\, \hat{j}^0(x) = i\int d^3x\; :\hat{\phi}^\dagger(x)\hat{\pi}^\dagger(x) - \hat{\pi}(x)\hat{\phi}(x): \; = \; \hat{N}_f - \hat{N}_g \;\equiv\; \hat{N}\,,$$

where the number operators for the $\hat{\phi}$ and $\hat{\phi}^\dagger$ are respectively defined as

$$\hat{N}_f \equiv \int d^3p \hat{N}_{f\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3}\, \hat{f}^\dagger_{\mathbf{p}}\hat{f}_{\mathbf{p}}\,, \qquad \hat{N}_g \equiv \int d^3p\, \hat{N}_{g\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3}\, \hat{g}^\dagger_{\mathbf{p}}\hat{g}_{\mathbf{p}}\,.$$

# Free charged scalar field

(See Chapter 6, Sections 6.2.7)

**Example of electric charge:** The conserved charge can be associated with electric charge when the charged scalar field is coupled to the electromagnetic field. In that case if $\phi$ has electric charge $q$ then $\phi^\dagger$ has charge $-q$ and we redefine the charge operator as $Q = q\hat{N} = q\hat{N}_f - q\hat{N}_g$. Then the total charge is the charge of the particles associated with the $\phi$ field minus the charge of the antiparticles associated with the $\phi^\dagger$ field. *In general particles and antiparticles will have opposite charges.*

**Conserved four-momentum:** We can similarly show that the normal-ordered four-momentum operator is

$$\hat{P}^\mu = (\hat{H}, \hat{\mathbf{P}}) = \int d^3p \, p^\mu [\hat{N}_{f\mathbf{p}} + \hat{N}_{g\mathbf{p}}] = \int (d^3p/(2\pi)^3) \, p^\mu [\hat{f}^\dagger_{\mathbf{p}} \hat{f}_{\mathbf{p}} + \hat{g}^\dagger_{\mathbf{p}} \hat{g}_{\mathbf{p}}]$$

and that $[\hat{N}_{f\mathbf{p}}, \hat{N}_{f\mathbf{p}'}] = [\hat{N}_{f\mathbf{p}}, \hat{N}_{g\mathbf{p}'}] = [\hat{N}_{g\mathbf{p}}, \hat{N}_{g\mathbf{p}'}] = 0$ . It then follows that $[\hat{P}^\mu, \hat{H}] = [\hat{P}^\mu, \hat{Q}] = 0$ and so, as expected, the system is invariant under spacetime translations and the charge $\hat{Q}$ is a conserved charge as it had to be since $[\hat{H}, \hat{Q}] = 0$.

**Generating functional for complex scalar field:** For the complex scalar field we have

$$Z[j, j^*] = Z[j_1]Z[j_2] = \lim_{T\to\infty(1-i\epsilon)} \frac{\int (\mathcal{D}\phi_1 \mathcal{D}\phi_2) \, e^{i(S[\phi_1, j_1] + S[\phi_2, j_2])}}{\int (\mathcal{D}\phi_1 \mathcal{D}\phi_2) \, e^{i(S[\phi_1] + S[\phi_2])}}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\int (\mathcal{D}\phi \mathcal{D}\phi^*) \, e^{iS[\phi, \phi^*, j, j^*]}}{\int (\mathcal{D}\phi \mathcal{D}\phi^*) \, e^{iS[\phi, \phi^*]}} .$$

# Free charged scalar field

(See Chapter 6, Sections 6.2.7)

**Free complex scalar field:** In the free case the action is a again quadratic in the fields and we can perform the gaussian integration to find

$$Z[j,j^*] = Z[j_1]Z[j_2] = \exp\left\{-\tfrac{1}{2}\int d^4x\, d^4y\, \left[j_1(x)\, D_F(x-y) j_1(y) + j_2(x)\, D_F(x-y) j_2(y)\right]\right\}$$
$$= \exp\left\{-\int d^4x\, d^4y\, j^*(x)\, D_F(x-y) j(y)\right\}\,,$$

which leads to the important results that

$$\langle 0|T\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle = (-i)^2 \frac{\delta^2}{\delta j^*(x)\,\delta j(y)} Z[j,j^*]\Big|_{j=0}$$
$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\int (\mathscr{D}\phi\mathscr{D}\phi^*)\, \phi(x)\phi^*(y)\, e^{iS[\phi,\phi^*]}}{\int (\mathscr{D}\phi\mathscr{D}\phi^*)\, e^{iS[\phi,\phi^*]}} = D_F(x-y)\,,$$
$$\langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \langle 0|T\hat{\phi}^\dagger(x)\hat{\phi}^\dagger(y)|0\rangle = 0\,.$$

Here $D_F(x-y)$ is the same Feynman propagator that we encountered in the hermitian scalar field case.