Lecture 2: Relativistic classical Field Theory and Relativistic Quantum Mechanics

Relativistic classical scalar field theories

(See Chapter 3, Sec 3.1)

For a real classical scalar field we identify one real degree of freedom with each point in three-dimensional space $\mathbf{x}$. So we have the correspondences

$$j = 1,2,\ldots,N \to \mathbf{x} \in \mathbb{R}^3, \quad \sum_{j=1}^N \to \int d^3x, \quad q_j(t) \to \phi(ct,\mathbf{x}) = \phi(x),$$

$$L(\vec{q}(t),\dot{\vec{q}}(t),t) \to L\big(\phi(x),\partial_\mu\phi(x),x\big) \equiv \int d^3x\, \mathscr{L}\big(\phi(x),\partial_\mu\phi(x),x\big), \text{ and}$$

$$S[\vec{q}\,] = \int dt\, L \to S[\phi] = \int dt\, L = \int dt \left(\int d^3x\, \mathscr{L}\right) = (1/c)\int d^4x\, \mathscr{L},$$

where $\mathscr{L}$ is called the *Lagrangian density*.

We will always choose a Lorentz invariant $\mathscr{L}$ so that a Lorentz covariant theory will result. The simplest such choice corresponds to a free scalar field and is

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 = \tfrac{1}{2}\partial_0\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2.$$

For *any* Lorentz invariant $\mathscr{L}$ the action $S[\phi]$ is also Lorentz invariant since the time dilation of dt cancels the length contraction in d^3x, which means that d^4x is Lorentz invariant.

Let us again impose Hamilton's principle, which in this context has the functional derivative form

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0\,.$$

Consider some arbitrary spacetime region R and consider some infinitesimal variation of the field ϕ, $\phi(x) \to \phi(x) + \epsilon r(x)$, where we choose $r(x)$ to vanish on the three-dimensional hypersurface S_R of R. By definition the functional derivative is

$$\int_R d^4x\, \frac{\delta S[\phi]}{\delta\phi(x)}\, r(x) \equiv \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{S[\phi+\epsilon r] - S[\phi]\right\}.$$

Relativistic classical scalar field theories

(See Chapter 3, Sec 3.1)

So Hamilton's principle leads to the result that

$$\begin{aligned}
0 &= \int_R d^4x\, \frac{\delta S[\phi]}{\delta\phi(x)}\, r(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{S[\phi+\epsilon r]-S[\phi]\right\} \\
&= \frac{1}{c}\int_R d^4x\, \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\{\mathscr{L}(\phi+\epsilon r,\partial_\mu\phi+\epsilon\partial_\mu r,x)-\mathscr{L}(\phi,\partial_\mu\phi,x)\right\} \\
&= \frac{1}{c}\int_R d^4x\, \left\{\frac{\partial\mathscr{L}}{\partial\phi(x)}r(x)+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)(x)}\partial_\mu r(x)\right\} \\
&= \frac{1}{c}\int_R d^4x\, \left\{\frac{\partial\mathscr{L}}{\partial\phi(x)}r(x)-\left(\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)(x)}\right)r(x)+\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)(x)}r(x)\right)\right\} \\
&= \frac{1}{c}\int_R d^4x\, \left\{\frac{\partial\mathscr{L}}{\partial\phi(x)}-\left(\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)(x)}\right)\right\}r(x)+\int_{S_R}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)(x)}\right)r(x)ds_\mu \\
&= \frac{1}{c}\int_R d^4x\, \left\{\frac{\partial\mathscr{L}}{\partial\phi(x)}-\left(\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)(x)}\right)\right\}r(x)\,,
\end{aligned}$$

where in the above we have used the fact that the surface contribution vanishes and we have used the Minkowski-space form of the divergence theorem

$$\int_R \partial_\mu F^\mu\, d^4x = \int_{S_R} F^\mu ds_\mu\,.$$

Here ds_μ is a four-vector infinitesimal surface element normal to the surface S_R of the spacetime region R and pointing out of the region for spacelike surfaces and into the region for timelike surfaces.

Relativistic classical scalar field theories

(See Chapter 3, Sec 3.1)

Since $r(x)$ and the space-time region R are arbitrary then we have arrived at the *Euler-Lagrange equations* for the scalar field $\phi(x)$. Generalizing to N scalar fields $\vec{\phi}(x) = \big(\phi_1(x), \ldots, \phi_N(x)\big)$ gives

$$S[\vec{\phi}\,] \equiv (1/c)\int_R d^4x\, \mathscr{L}(\vec{\phi}, \partial_\mu \vec{\phi}\,)\,,$$

$$0 = c\frac{\delta S[\vec{\phi}\,]}{\delta \phi_j(x)} = \frac{\partial \mathscr{L}}{\partial \phi_j(x)} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_j)(x)} \right) \quad \text{for } j = 1,2,\ldots,N.$$

Hamiltonian formulation: By analogy with classical mechanics, for every $\phi(ct, \mathbf{x}) = \phi(x)$ we define the corresponding conjugate momentum $\pi(ct, \mathbf{x}) = \pi(x)$ as

$$\pi(x) \equiv c\frac{\delta L}{\delta \dot{\phi}(x)} = \frac{\delta L}{\delta(\partial_0 \phi)(x)} = \frac{\partial \mathscr{L}}{\partial(\partial_0 \phi)(x)}\,.$$

Provided that the Hessian `matrix' $M_L(x,y)\,|_{x^0=y^0} = \delta^2 \mathscr{L}/\delta(\partial_0\phi(x))\delta(\partial_0\phi(y))\,|_{x^0=y^0}$ is positive definite, then we can perform a Legendre transform to replace $\partial_0\phi(x)$ with $\pi(x)$ and define the Hamiltonian as

$$H \equiv [\,\int d^3x\, \pi(x)\partial_0\phi(x)] - L = \int d^3x\, [\pi(x)\partial_0\phi(x) - \mathscr{L}] \equiv \int d^3x\, \mathscr{H}(\phi, \nabla\phi, \pi, x)\,,$$

where $\mathscr{H}(\phi, \nabla\phi, \pi, x)$ is called the *Hamiltonian density*.

It is not difficult to show that Hamilton's equations take the form

$$\partial_0 \phi = \frac{\partial \mathscr{H}}{\partial \pi}\,, \quad \partial_0 \pi = -\frac{\partial \mathscr{H}}{\partial \phi} + \nabla \cdot \frac{\partial \mathscr{H}}{\partial(\nabla \phi)} \quad \text{and} \quad \partial_\mu^{\text{ex}} \mathscr{H} = -\,\partial_\mu^{\text{ex}} \mathscr{L}\,,$$

where ∂_μ^{ex} acts only on the *explicit* x-dependence in $\mathscr{L}(\phi(x), \partial_\mu\phi(x), x)$.

Poisson bracket in classical field theory: Recall that the Poisson bracket always involves quantities at equal times. Their field theory form replaces partial derivatives with functional derivatives and so

$$\{F, G\} \equiv \int d^3z \left(\frac{\delta F}{\delta \vec{\phi}(x^0, \mathbf{z})} \cdot \frac{\delta G}{\delta \vec{\pi}(x^0, \mathbf{z})} - \frac{\delta F}{\delta \vec{\pi}(x^0, \mathbf{z})} \cdot \frac{\delta G}{\delta \vec{\phi}(x^0, \mathbf{z})} \right).$$

Relativistic classical scalar field theories

(See Chapter 3, Sec 3.1)

Frequent use is made of the functional derivative results

$$\frac{\delta\phi_i(x^0,\mathbf{x})}{\delta\phi_j(x^0,\mathbf{y})} = \delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \quad \text{and} \quad \frac{\delta\nabla\phi_i(x^0,\mathbf{x})}{\delta\phi_j(x^0,\mathbf{y})} = \delta_{ij}\nabla^x\delta^3(\mathbf{x}-\mathbf{y})\ .$$

Similar to the discrete mechanics case we find

$$\partial_0 F = \{F,H\} + \partial_0^{\mathrm{ex}} F\ .$$

Self-interacting scalar field: Adding a self interaction $U(\phi)$ leads to

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu\phi\right)^2 - \tfrac{1}{2}m^2\phi^2 - U(\phi) = \tfrac{1}{2}\partial_0\phi^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2 - U(\phi)$$

and the Euler-Lagrange equations

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi(x) + \partial U(\phi)/\partial\phi(x) = 0\ .$$

We can write the Hamiltonian as

$$H = \int d^3x\,\mathscr{H} = \int d^3x\,[\tfrac{1}{2}\pi(x)^2 + \tfrac{1}{2}\left(\nabla\phi(x)\right)^2 + \tfrac{1}{2}m^2\phi(x)^2 + U(\phi)] \equiv T[\pi] + V[\phi]\ ,$$

where $V[\phi] \equiv \int d^3x \left[\tfrac{1}{2}\left(\nabla\phi(x)\right)^2 + \tfrac{1}{2}m^2\phi(x)^2 + U(\phi)\right]$.

Free scalar field: For a free scalar field $U(\phi) = 0$ and the Euler-Lagrange equations are

$(\Box + m^2)\phi(x) = (\partial_\mu\partial^\mu + m^2)\phi(x) = 0$. - *Klein-Gordon equation* for classical field.

Noether's Theorem for Classical Fields

(See Chapter 3, Sec 3.2)

Noether currents: Consider an infinitesimal transformation of both the coordinates and fields $\overrightarrow{\phi}(x) = \left(\phi_1(x), \ldots, \phi_N(x)\right)$, such that

$$x^\mu \to x'^\mu \equiv x^\mu + \delta x^\mu(x) \equiv x^\mu + d\alpha X^\mu(x) \qquad \text{and}$$

$$\phi_i(x) \to \phi_i'(x') \equiv \phi_i(x) + \delta\phi_i(x) \equiv \phi_i(x) + d\alpha \Phi_i(x) \,,$$

which *defines* $X^\mu(x)$ and $\Phi_i(x)$. If the action is form invariant, $S[\phi'] = S[\phi]$, under such a transformation then we have a *symmetry of the theory*. This symmetry leads to a conserved four-vector current called a *conserved Noether current* (named after Emmy Noether),

$$j^\mu(x) \equiv \vec{\pi}^\mu(x) \cdot \left[\overrightarrow{\Phi}(x) - [\partial_\nu \overrightarrow{\phi}(x)] X^\nu(x)\right] + \mathscr{L}(\overrightarrow{\phi}, \partial_\mu \overrightarrow{\phi}, x) X^\mu(x) \,, \qquad \text{where} \qquad \partial_\mu j^\mu(x) = 0 \,.$$

In this current we have made the definition

$$\vec{\pi}^\mu(x) \equiv \partial\mathscr{L}/\partial(\partial_\mu \overrightarrow{\phi})(x) \,,$$

where recall that the conjugate momentum is $\pi(x) \equiv \pi^0(x)$. For details see Sec. 3.2. We can always redefine a current by multiplying by a constant a and by adding the four-divergence of an antisymmetric tensor $A^{\mu\nu} = -A^{\nu\mu}$, since if $\partial_\mu j^\mu = 0$ then for any $j'^\mu \equiv a j^\mu + \partial_\nu A^{\nu\mu}$ we have $\partial_\mu j'^\mu = 0$.

Conserved charge: For each conserved current $j^\mu(x) \equiv \left(j^0(x), \mathbf{j}(x)\right) \equiv \left(c\rho(x), \mathbf{j}(x)\right)$ there is a conserved charge, Q,which is the spatial integral of the charge density $\rho(x)$,

$$Q \equiv \int d^3x\, \rho(x) = (1/c) \int d^3x\, j^0(x) \,.$$

Consider some three-dimensional region R bounded by the two-dimensional surface S_R. If the total current density passing through the surface vanishes, we then find

$$dQ/dt = \int_R d^3x\, \partial_0 j^0(x) = \int_R d^3x\, \nabla \cdot \vec{j}(x) = \int_{S_R} d\vec{s} \cdot \vec{j}(x) = 0$$

where we have used the three-dimensional form of the divergence theorem to perform the last step.

Noether's Theorem for Classical Fields

(See Chapter 3, Sec 3.2)

Example: Consider the case of a complex scalar field $\phi(x) = [\phi_1(x) + i\phi_2(x)]/\sqrt{2}$, where $\phi_1, \phi_2 \in \mathbb{R}$ and with the Lagrangian density

$$\mathscr{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 = (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2\phi^*\phi = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 - m^2(\phi_1)^2 - m^2(\phi_2)^2\right].$$

This $\mathscr{L}$ is invariant under the global phase transformation $\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)$ and $\phi^*(x) \to \phi^{*'}(x) = e^{-i\alpha}\phi(x)$. So the action $S[\phi, \phi^*]$ is invariant and this is a symmetry of the theory. Let $d\alpha$ be infinitesimal and work to $\mathcal{O}(d\alpha)$. Note that there is no transformation of x^μ and so $X^\mu(x) = 0$, whereas we recognize that $\Phi(x) = i\phi(x)$ and $\Phi^*(x) = -i\phi^*(x)$. The conserved Noether current is then

$$j^\mu \equiv \vec{\pi}^\mu \cdot \left[\overrightarrow{\Phi} - [\partial_\nu \overrightarrow{\phi}\,]X^\nu\right] + \mathscr{L}X^\mu \to \pi^\mu(x)\Phi(x) + \pi^*(x)\Phi^*(x) = i\left[\phi(\partial^\mu\phi^*) - \phi^*(\partial^\mu\phi)\right],$$

since $\pi^\mu(x) = \partial^\mu\phi^*(x)$ and $\pi^{\mu *}(x) = \partial^\mu\phi(x)$. A complex scalar field is then a charged scalar field in the sense that is has a conserved charge $Q = \int d^3x\, \rho(x)$. We will later couple it to an electromagnetic field A^μ with minimal coupling, which leads to a $U(1)$ gauge theory where Q is the electric charge.

Stress-Energy Tensor

(See Chapter 3, Sec 3.2.2)

Translationally invariant systems: If $\mathscr{L}$ has no explicit spacetime dependence $\mathscr{L}(\overrightarrow{\phi}, \partial_\mu \overrightarrow{\phi}, x) \to \mathscr{L}(\overrightarrow{\phi}, \partial_\mu \overrightarrow{\phi})$ then the action $S[\overrightarrow{\phi}]$ is form invariant under spacetime translations, i.e., then $x^\mu \to x'^\mu = x^\mu + a^\mu$ for some constants $a^\mu \in \mathbb{R}$ is a symmetry of the system. Under an infinitesimal translation $d\alpha \to da^\nu$ and so there will be *four* conserved currents and so $X^\mu \to X^\mu{}_\nu$ and $\Phi \to \Phi_\nu$. We have $x^\mu \to x^\mu + da^\mu = x^\mu + da^\nu \delta^\mu{}_\nu \equiv x^\mu + da^\nu X^\mu{}_\nu$ and so $X^\mu{}_\nu = \delta^\mu{}_\nu$. There is no transformation of the fields and so $\Phi_\nu(x) = 0$. The four conserved currents are then given by $(j^\mu)_\nu \equiv T^\mu{}_\nu$, where we are free to insert a conventional negative sign into the current

$$T^\mu{}_\nu \equiv (j^\mu)_\nu = -\vec{\pi}^\mu \cdot \left[\overrightarrow{\Phi}_\nu - [\partial_\rho \overrightarrow{\phi}] X^\rho{}_\nu(x)\right] - \mathscr{L} X^\mu{}_\nu = \vec{\pi}^\mu \cdot \partial_\nu \overrightarrow{\phi} - \delta^\mu{}_\nu \mathscr{L} \quad \text{with} \quad \partial_\mu T^{\mu\nu}(x) = 0 \,.$$

We refer to $T^{\mu\nu}$ as the *stress-energy tensor*. Typically $T^{\mu\nu}$ will not be symmetric.

With four conserved currents there will be four conserved charges, which we can define as

$$P^0 = P_0 = (1/c)\int d^3x\, T^0{}_0 = \int d^3x\, (\mathscr{H}/c) = (H/c)\,,$$

$$P^i = -P_i = -(1/c)\int d^3x\, T^0{}_i = -(1/c)\int d^3x\, \vec{\pi} \cdot \partial_i \overrightarrow{\phi}\,.$$

The four conserved charges are the total four-momentum of the fields $P^\mu = (P^0, \mathbf{P}) = (H/c, \mathbf{P})$, where H is the classical Hamiltonian for the scalar fields.

The energy density for the system is

$$u \equiv T^{00} = \mathscr{H}$$

and the physical three-momentum density is given by

$$p^i_{\text{den}} = T^{0i}/c = \vec{\pi} \cdot \partial^i \overrightarrow{\phi}/c = -T^0{}_i/c = -\vec{\pi} \cdot \partial_i \overrightarrow{\phi}/c\,.$$

Angular Momentum Tensor

(See Chapter 3, Sec 3.2.3)

Poincaré invariant scalar field systems: If $\mathscr{L}$ is a Lorentz scalar, as it must be in a Lorentz covariant system, then $S[\overrightarrow{\phi}]$ is by construction form invariant under Lorentz transformations. We assume that $\overrightarrow{\phi} = (\phi_1, \ldots, \phi_N)$ are scalar fields and so have no intrinsic spin ($s = 0$). For an inifinitessimal Lorentz transformation $x^\mu \to x'^\mu = x^\mu + d\omega^{\mu\nu} x_\nu$ with $d\omega^{\mu\nu} = -\, d\omega^{\nu\mu}$. There are six independent ω's and so there will be six independent Noether currents. We can write

$$x^\mu \to x'^\mu = x^\mu + d\omega^{\mu\nu} x_\nu \equiv x^\mu + \tfrac{1}{2} d\omega^{\rho\sigma} X^\mu{}_{\rho\sigma}\,, \text{ where we have defined } X^\mu{}_{\rho\sigma} \equiv \left(\delta^\mu{}_\rho x_\sigma - \delta^\mu{}_\sigma x_\rho\right).$$

Since the scalar fields do not change then $\Phi_{\rho\sigma} = 0$. The *six independent Noether currents* are then

$$\mathscr{J}^\mu{}_{\rho\sigma}(x) = \left(-\vec{\pi}^\mu \cdot \partial_\nu \overrightarrow{\phi} + \delta^\mu{}_\nu \mathscr{L}\right) X^\nu{}_{\rho\sigma} = -\, T^\mu{}_\nu X^\nu{}_{\rho\sigma} = x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho\,,$$

where $\quad \mathscr{J}^\mu{}_{\rho\sigma} = -\, \mathscr{J}^\mu{}_{\sigma\rho} \quad$ and $\quad \partial_\mu \mathscr{J}^\mu{}_{\rho\sigma} = 0\,.$

The six independent conserved charges are

$$M^{\rho\sigma} = (1/c)\int d^3x\, \mathscr{J}^{0\rho\sigma} = (1/c)\int d^3x\, (x^\rho T^{0\sigma} - x^\sigma T^{0\rho}) = \int d^3x\, (x^\rho p^\sigma_{\text{dens}} - x^\sigma p^\rho_{\text{dens}})\,.$$

This is the classical field theory analog of the relativistic particle result that $M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ from Lecture 1, where we identified $J^i = L^i = (\mathbf{x} \times \mathbf{P})^i = \epsilon^{ijk} x^j P^k = \frac{1}{2}\epsilon^{ijk} M^{jk}$ for a spinless particle. So the three conserved charges associated with the *angular momentum of spinless fields*, $\mathbf{J} = \mathbf{L}$, are

$$J^i = \tfrac{1}{2}\epsilon^{ijk} M^{jk} = \int d^3x\, \tfrac{1}{2}\epsilon^{ijk}\left[x^j (T^{0k}/c) - x_k (T^{0j}/c)\right] = \int d^3x\, \left(\mathbf{x} \times \mathbf{p}_{\text{den}}\right)^i \equiv L^i\,.$$

The other three conserved charges are $K^i = M^{0i}/c$ and are referred to as the *dynamic mass moment* of the fields and can be written as $\mathbf{K} = c\left[t\mathbf{P} - \gamma(V) M \mathbf{X}_{\text{com}}\right]$, where M is the total mass of the fields (total energy when $\mathbf{P} = 0$), $\mathbf{X}_{\text{com}}$ is the center-of-mass (c.o.m) of the fields, $\mathbf{V} \equiv \mathbf{P}/\gamma(\mathbf{V})\mathbf{M}$ and $\mathbf{P}$ is the total three-momentum. Since $d\mathbf{P}/dt = 0$ then $d\mathbf{V}/dt = 0$. So $d\mathbf{K}/dt = 0 \implies \mathbf{V} = d\mathbf{X}_{\text{com}}/dt$. So the total system behaves as a relativistic particle with internal structure and total "spin" $S = J_{\mathbf{P}=0} = L_{\mathbf{P}=0}\,.$

Intrinsic Angular Momentum
(See Chapter 3, Sec 3.2.4)

Poincaré invariant systems with intrinsic spin: Consider a set of fields that transform not as a set of scalars but in a nontrivial way under Lorentz transformations such that

$$\vec{\phi}(x) \to \vec{\phi}'(x') = S\vec{\phi}(x) \quad \text{with} \quad x' = \Lambda x \quad \text{or in component form}$$

$$\phi_r(x) \to \phi_r'(x') = S_{rs}\phi_s(x) \quad \text{with} \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

where where $r, s = 1,\ldots,N$, where summation over repeated indices is to be understood, and where $S \equiv S[\Lambda]$ belongs to an $N \times N$ matrix representation of the Lorentz group. We can define

$$S_{rs} \equiv \delta_{rs} + \frac{1}{2} d\omega^{\rho\sigma}(\Sigma_{rs})_{\rho\sigma} \quad \text{with} \quad (\Sigma_{rs})_{\rho\sigma} = -(\Sigma_{rs})_{\sigma\rho}\,.$$

$$\phi_r'(x') = S_{rs}\phi_s(x) = \phi_r(x) + \frac{1}{2} d\omega^{\rho\sigma}(\Sigma_{rs})_{\rho\sigma}\phi_s(x) \equiv \phi_r(x) + \frac{1}{2} d\omega^{\rho\sigma}\Phi_{r\rho\sigma}(x) \quad \text{with}$$

$$\Phi_{r\rho\sigma}(x) \equiv (\Sigma_{rs})_{\rho\sigma}\phi_s(x)$$

We also have for a Lorentz transformation as before $x^\mu \to x'^\mu = x^\mu + d\omega^{\mu\nu}x_\nu \equiv x^\mu + \frac{1}{2} d\omega^{\rho\sigma} X^\mu{}_{\rho\sigma}$. So then the six independent conserved Noether currents have the form

$$\mathcal{J}^\mu{}_{\rho\sigma} = (-\pi_r^\mu \partial_\nu \phi_r + \delta^\mu{}_\nu \mathcal{L}) X^\nu{}_{\rho\sigma} + \pi_r^\mu \Phi_{r\rho\sigma} = (x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho) + \pi_r^\mu (\Sigma_{rs})_{\rho\sigma}\phi_s \equiv (x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho) + R^\mu{}_{\rho\sigma}$$

where $\partial_\mu \mathcal{J}^\mu{}_{\rho\sigma} = 0$ and where we have defined $R^\mu{}_{\rho\sigma}$. The six corresponding conserved charges are

$$M^{\rho\sigma} = (1/c)\int d^3x\, \mathcal{J}^{0\rho\sigma} = (1/c)\int d^3x\,[(x^\rho T^{0\sigma} - x^\sigma T^{0\rho}) + \pi_r(\Sigma_{rs})^{\rho\sigma}\phi_s] \equiv L^{\rho\sigma} + S^{\rho\sigma}\,,$$

We define the orbital ($L^{\rho\sigma}$) and intrinsic ($S^{\rho\sigma}$) angular momentum components of $M^{\rho\sigma}$ respectively as

$$L^{\rho\sigma} \equiv (1/c)\int d^3x\, \left(x^\rho T^{0\sigma} - x^\sigma T^{0\rho}\right) = \int d^3x\, \left(x^\rho p^\sigma_{\rm dens} - x^\sigma p^\rho_{\rm dens}\right),$$

$$\Sigma^{\rho\sigma} \equiv \int d^3x\, \left(\pi_r(\Sigma_{rs})^{\rho\sigma}\phi_s/c\right).$$

We can similarly decompose the total angular momentum $(\mathbf{J})$ carried by the classical fields into total orbital $(\mathbf{L})$ and total intrinsic angular momentum $(\mathbf{S})$ components as

$$\mathbf{J} = \mathbf{L} + \mathbf{S}\,, \text{ where } J^i \equiv \frac{1}{2}\epsilon^{ijk}M^{jk},\ S^i \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^{jk} \quad \text{and} \quad L^i \equiv \frac{1}{2}\epsilon^{ijk}L^{jk} = \int d^3x\,(\mathbf{x} \times \mathbf{p}_{\rm den})^i\,.$$

Internal Symmetries

(See Chapter 3, Sec 3.2.5)

Internal symmetries of the fields: Consider a symmetry where the action is invariant under transformation of the field components between themselves at *the same spacetime point*, i.e.,

$$\phi_r(x) \to \phi'_r(x) = R_{rs}\phi_s(x) .$$

We refer to this as an *internal symmetry*. Note that since $x^\mu \to x'^\mu = x^\mu$ then $X^\mu(x) = 0$.

We will typically be interested in situations where the matrix R is any arbitrary element of an $N \times N$ matrix representation of some group, e.g, the $N \times N$ representation of the group $SU(N)$. We then say that the action is invariant under this group of internal transformations. For an infinitesimal transformation R we define λ_{rs} using

$$R_{rs} \equiv \delta_{rs} + d\alpha\lambda_{rs} ,$$

which leads to the definition of $\Phi_r(x)$,

$$\phi'_r(x) = \phi_r(x) + d\alpha\lambda_{rs}\phi_s(x) \equiv \phi_r(x) + d\alpha\Phi_r(x) .$$

The conserved Noether current is then

$$j^\mu(x) = \vec{\pi}^\mu(x) \cdot \overrightarrow{\Phi}(x) = \pi^\mu_r(x)\lambda_{rs}\phi_s(x) \quad \text{where} \quad \partial_\mu j^\mu(x) = 0 .$$

The conserved charge associated with this internal symmetry is

$$Q = \int d^3x\, \rho(x) = \int d^3x\, j^0(x)/c = \int d^3x\, \left(\pi_r(x)\lambda_{rs}\phi_s(x)/c\right) ,$$

where recall that $\pi_r \equiv \pi^0_r$.

Belinfante-Rosenfeld Tensor

(See Chapter 3, Sec 3.2.6)

Making a symmetric stress-energy tensor: As noted earlier the "canonical" stress-energy tensor $T^{\mu\nu}$ that we have defined is not in general symmetric. However, we can use freedoms to redefine conserved currents so that the so-called Belinfante-Rosenfeld tensor $\bar{T}^{\mu\nu}$ is a symmetric form of $T^{\mu\nu}$. It is $\bar{T}^{\mu\nu}$ that induces the curvature of spacetime in general relativity.

In a Poincaré invariant system we have from translational invariance that $\partial_\mu T^{\mu\nu} = 0$ and from Lorentz invariance that $\partial_\mu \mathcal{J}^\mu{}_{\rho\sigma} = 0$. Then we find

$$0 = \partial_\mu \mathcal{J}^\mu{}_{\rho\sigma} = \partial_\mu \Big((x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho) + \pi^\mu_r (\Sigma_{rs})_{\rho\sigma} \phi_s \Big) = g_{\mu\rho} T^\mu{}_\sigma - g_{\mu\sigma} T^\mu{}_\rho + \partial_\mu R^\mu{}_{\rho\sigma} \,.$$

Recall that Recall that $(\Sigma_{rs})_{\rho\sigma} = -(\Sigma_{rs})_{\sigma\rho}$ and so $R^{\mu\rho\sigma} = -R^{\mu\sigma\rho}$. The antisymmetric part of the stress-energy tensor is $\frac{1}{2}\left(T^{\rho\sigma} - T^{\sigma\rho}\right) = -\frac{1}{2}\partial_\mu R^{\mu\rho\sigma}$ and so $T^{\mu\nu}$ is only nonsymmetric when the classical field has intrinsic angular momentum, $\Sigma_{rs} \neq 0$. If we define

$$K^{\mu\rho\sigma} \equiv \tfrac{1}{2}(R^{\mu\rho\sigma} + R^{\rho\sigma\mu} + R^{\sigma\rho\mu}) \qquad \text{and} \qquad \bar{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$$

then it can be shown that (see Sec. 3.2.6)

$$\partial_\mu \bar{T}^{\mu\nu} = 0 \qquad \text{and} \qquad \bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}$$

as required. Since $K^{\rho\mu\nu} = -K^{\mu\rho\nu}$ we see that we have modified $T^{\mu\nu}$ by the allowed addition of the four-divergence of an antisymmetric tensor to give $\bar{T}^{\mu\nu}$. It is readily shown that $\bar{P}^\mu = (1/c)\int d^3x\, \bar{T}^{0\nu} = P^\mu$.

Similarly defining $\bar{\mathcal{J}}^\mu{}_{\rho\sigma} \equiv \left(x_\rho \bar{T}^\mu{}_\sigma - x_\sigma \bar{T}^\mu{}_\rho \right)$ we find that $\bar{\mathcal{J}}^{\mu\rho\sigma} = \mathcal{J}^{\mu\rho\sigma} + \partial_\lambda \left(x^\rho K^{\lambda\mu\sigma} - x^\sigma K^{\lambda\mu\rho} \right)$ where since $K^{\rho\mu\sigma} = -K^{\mu\rho\sigma}$ we have $\partial_\mu \bar{\mathcal{J}}^\mu{}_{\rho\sigma} = \partial_\mu \mathcal{J}^\mu{}_{\rho\sigma} = 0$ and $\bar{M}^{\rho\sigma} = (1/c)\int d^3x\, \bar{\mathcal{J}}^{0\rho\sigma} = M^{\rho\sigma}$.

Poincaré Lie Algebra and Poisson Brackets

(See Chapter 3, Sec 3.2.7 and 3.2.8)

In classical mechanics a conserved charge Q generates a canonical transformation that leaves the system invariant. So it is also true in classical field theory that a conserved charge Q satisfies

$$dQ/dt = \{H, Q\} = 0$$

and generates a canonical transformation on the fields (the "coordinates") ϕ_i and their conjugate momenta $\pi_i = \pi_i^0$ that is the corresponding infinitesimal symmetry transformation,

$$d\phi_i(x) = d\alpha\{\phi_i(x), Q\} = d\alpha\, \delta Q/\delta\pi_i(x)\,, \qquad d\pi_i(x) = d\alpha\{\pi_i(x), Q\} = -\,d\alpha\, \delta Q/\delta\phi_i(x)\,.$$

We expect that the conserved charges P^μ and $M^{\mu\nu}$ are the generators of the translations and the Lorentz transformations respectively. This can be explicitly confirmed since we can show that

$$\{\vec{\phi}(x), P^\mu\} = \partial^\mu \vec{\phi}(x) \qquad \text{and} \qquad \{\vec{\phi}(x), M^{\mu\nu}\} = \vec{\Psi}^{\mu\nu}(x) = \Sigma_{\rho\sigma}\vec{\phi}(x) + x_\rho\partial_\sigma\vec{\phi}(x) - x_\sigma\partial_\rho\vec{\phi}(x)\,,$$

which correspond exactly to the translations and Lorentz transformations respectively [See Eqs. (3.2.51) and (3.2.92)]. With some effort it can be shown that these generators P^μ and $M^{\mu\nu}$ satisfy the Lie algebra of the Poincaré group where the role of the *Lie bracket* is played by the the Poisson bracket $\{\cdots,\cdots\}$ rather that the commutator $[\cdots,\cdots]$,

$$i\{P^\mu, P^\nu\} = 0\,, \qquad i\{P^\mu, M^{\rho\sigma}\} = i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho)\,,$$

$$i\{M^{\mu\nu}, M^{\rho\sigma}\} = i\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}\right)\,.$$

Now we can appreciate the beauty of Dirac's canonical quantization approach, which results in the replacement of the Poisson bracket (or Dirac bracket for singular systems) with the commutator in the quantized version of the theory. This automatically preserves all of the dynamics of the classical theory into the quantum theory, i.e., *all Poisson (or Dirac) brackets are replaced by commutators and the quantum operators obey the same equations of motion as their classical counterparts*.

The exception to this rule occurs when we have to renormalize the quantum theory and when every possible regularization of the theory violates some classical relation. Such an unavoidable violation is called a *quantum anomaly*.

Classical Electromagnetic Field

(See Chapter 3, Sec 3.3)

A detailed review of Maxwell's equations and the covariant formulation of electromagnetism and the associated need for gauge fixing of the four-vector field A^μ is given in Chapter 2, Sec. 2.7. We summarize here the formulation of electromagnetism as a relativistic classical field theory. In Sec. 2.7.1 we showed that Maxwell's equations can be succinctly written as

$\partial_\mu \tilde{F}^{\mu\nu} = 0$ and $\partial_\mu F^{\mu\nu} = (1/c) j^\nu$,

where where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\nabla\Phi - (\partial\mathbf{A}/\partial t)$, $j^\mu = (j^0, \mathbf{j}) = (\rho/c, \mathbf{j})$ and in Lorentz-Heaviside units $A^\mu = (A^0, \mathbf{A}) \equiv (\Phi, \mathbf{A})$. The result that $\partial_\mu \tilde{F}^{\mu\nu} = 0$ follows automatically from these definitions alone. All of the dynamics is contained in the equation of motion $\partial_\mu F^{\mu\nu} = (1/c) j^\nu$. There are very many A^μ corresponding to the same $\mathbf{E}$ and $\mathbf{B}$ fields, which means that we need to "gauge-fix" the A^μ to reduce the unphysical degrees of freedom. Common choices are: (i) *Coulomb gauge* where $\nabla \cdot \mathbf{A} = 0$; and (ii) *Lorenz gauge* where $\partial_\mu A^\mu = 0$.

Consider the action $S[A] = \int dt\, L = (1/c)\int d^4x\, \mathscr{L} = (1/c)\int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (1/c) j_\mu A^\mu\right]$

Applying Hamilton's principle to this action gives the Euler-Lagrange equations

$$0 = c\frac{\delta S[A]}{\delta A_\nu(x)} = \frac{\partial \mathscr{L}}{\partial A_\nu(x)} - \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)(x)}\right) = -(1/c) j^\nu(x) + \partial_\mu F^{\mu\nu}(x),$$

and so we recover Maxwell's equations. In the absence of an external current we have

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

which is Poincaré invariant. The canonical stress-energy tensor is

$$T^\mu{}_\nu(x) = \frac{\partial \mathscr{L}}{\partial(\partial_\mu A^\tau)}\partial_\nu A^\tau - \delta^\mu{}_\nu \mathscr{L} = -F^{\mu\tau}\partial_\nu A_\tau + \frac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau}, \quad \text{where} \quad \partial_\mu T^{\mu\nu} = 0.$$

Classical Electromagnetic Field

(See Chapter 3, Sec 3.3)

Since $(F^{\mu\tau}A_\nu)$ is antisymmetric under the interchange of μ and τ we can define a modified conserved stress-energy tensor by choosing $K^{\mu\tau\nu} = F^{\mu\tau}A^\nu$,

$$\bar{T}^\mu{}_\nu \equiv T^\mu{}_\nu + \partial_\tau\left(F^{\mu\tau}A_\nu\right) = T^\mu{}_\nu + F^{\mu\tau}\partial_\tau A_\nu = -F^{\mu\tau}F_{\nu\tau} + \tfrac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau} \;,$$

which satisfies $\bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}$ and $\partial_\mu \bar{T}^{\mu\nu} = 0$. $\bar{T}^{\mu\nu}$ is the Belinfante-Rosenfeld stress-energy tensor for the electromagnetic field. We find that

$$P^0 = E/c = (1/c)\int d^3x\, u = (1/c)\int d^3x\, \bar{T}^{00} = (1/c)\int d^3x\, \tfrac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \;,$$

$$P^i = \int d^3x\, p^i_{\text{dens}} = (1/c)\int d^3x\, \bar{T}^{0i} = \int d^3x\, S^i/c^2 = (1/c)\int d^3x\, (\mathbf{E}\times\mathbf{B})^i \;.$$

Similarly, for the improved angular momentum tensor $\overline{\mathcal{J}}^\mu{}_{\rho\sigma} \equiv \left(x_\rho \bar{T}^\mu{}_\sigma - x_\sigma \bar{T}^\mu{}_\rho\right)$ we have

$\partial_\mu \overline{\mathcal{J}}^\mu{}_{\rho\sigma} = \partial_\mu \mathcal{J}^\mu{}_{\rho\sigma} = 0$ and $\bar{M}^{\rho\sigma} = (1/c)\int d^3x\, \bar{\mathcal{J}}^{0\rho\sigma} = M^{\rho\sigma}$,

which leads to the result

$$J^i = \tfrac{1}{2}\epsilon^{ijk}M^{jk} = \int d^3x\, (\mathbf{x}\times\mathbf{p}_{\text{dens}})^i = (1/c)\int d^3x\, [\mathbf{x}\times(\mathbf{E}\times\mathbf{B})]^i \;.$$

Hamiltonian formulation of electromagnetism: This is too complex to discuss here because electromagnetism is a singular system, since $\pi^0(x) = \partial\mathcal{L}/\partial(\partial_0 A_0)(x) = 0$ and so the Hessian has zero eigenvalues. We need to use the DIrac-Bergmann algorithm to introduce constraints. This leads to the Dirac brackets that show how to canonically quantize the system. For a full discussion see Chapter 3, Sec. 3.3.2. This results in two constraints and so the four degrees of freedom in A^μ reduce to the two physical degrees freedom of the electromagnetic field. It is easiest to formulate the Hamiltonian approach in Coulomb gauge, which is what is done in Sec. 3.3.2. We generalize to other gauge choices later in these lectures (for details of this see Chapter 6, Sec 6.4.4).

Relativistic Quantum Mechanics (RQM)

(See Chapter 4, Secs 4.3 & 4.4)

Klein-Gordon equation: Let $\phi_{\mathbf{k}}(x)$ be a plane wave wavefunction, then we would expect that it is a simultaneous eigenstate of energy and three-momentum so that

$$\hat{H}\phi_{\mathbf{k}}(x) = i\hbar(\partial/\partial t)\phi_{\mathbf{k}}(x) = \hbar\omega\phi_{\mathbf{k}}(x) = E\phi_{\mathbf{k}}(x) \quad \text{and}$$

$$\hat{\mathbf{p}}\phi_{\mathbf{k}}(x) = -i\hbar\nabla\phi_{\mathbf{k}}(x) = \hbar\mathbf{k}\phi_{\mathbf{k}}(x) = \mathbf{p}\phi_{\mathbf{k}}(x) .$$

In order to describe the motion of a relativistic particle we expect that $p^2 = p^\mu p_\mu = (E/c)^2 - \mathbf{p}^2 = m^2c^2$ and so we require

$$\hat{p}^2\phi_{\mathbf{k}}(x) = \hat{p}^\mu\hat{p}_\mu\phi_{\mathbf{k}}(x) = \left((\hat{H}^2/c^2) - \hat{\mathbf{p}}^2\right)\phi_{\mathbf{k}}(x) = -\hbar^2\partial^\mu\partial_\mu\phi_{\mathbf{k}}(x) = m^2c^2\phi_{\mathbf{k}}(x) ,$$

which we recognize is just the *Klein-Gordon equation (KGE)* of relativistic classical field theory with appropriate factors of $\hbar$ and c to ensure that m has the physical dimension of mass. So we have

$$\left[\partial_\mu\partial^\mu + (mc/\hbar)^2\right]\phi = \left[\Box + (mc/\hbar)^2\right]\phi = 0 .$$

We know that this KGE can be obtained as the equation of motion for a Lagrangian density

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - (m/\hbar c)^2\phi^2\right) .$$

Note that since we have $\left(\hat{H}^2/c^2\right) - \hat{\mathbf{p}}^2)\phi_{\mathbf{k}}(x)$ then *both positive and negative energy solutions* are possible. Negative energy $\Longrightarrow$ antiparticles. *We always find (QM + special relativity) $\Longrightarrow$ antiparticles.*

The quantum mechanical wavefuction is complex and so it is natural to consider a complex scalar field $\phi(x)$. This has the Lagrangian density $\mathscr{L} = \partial_\mu\phi\partial^\mu\phi * - (m/\hbar c)^2\phi\phi*$, with KGE as the equation of motion, $\left[\partial_\mu\partial^\mu - (m/\hbar c)2\right]\phi = 0$, and with the conserved current $j^\mu = iq[\phi^*\partial^\mu\phi - (\partial^\mu\phi)^*\phi]$. Since the density ρ is *not positive definite* it cannot be interpreted as a probability density as we might expect from QM, rather we need to think of it as a *wave*. In the nonrelativistic limit we do recover the Schrödinger equation and the normal QM interpretation however, [see Eq. (4.2.21)].

Relativistic Quantum Mechanics (RQM)

(See Chapter 4, Secs 4.3 & 4.4)

Spinless charged particle interacting with e.m. field: As seen earlier the minimal coupling to the e.m. field (including factors of $\hbar$ and c to get dimensions correct) is given by $\partial_\mu \to D_\mu \equiv \partial_\mu + i(q/\hbar c)A_\mu$. The Lagrangian density then becomes $\mathscr{L} = D_\mu \phi D^\mu \phi^* - (m/\hbar c)^2 \phi\phi^*$, with the modified KGE $\left[D_\mu D^\mu - (m/\hbar c)2\right]\phi = 0$ and the conserved current $j^\mu = iq\left[\phi^* D^\mu \phi - \phi(D^\mu\phi)^*\right]$. In the nonrelativistic limit we recover the usual QM Schrödinger equation for a (spinless) particle interacting with an e.m. field, i.e., $\hat{H}^{\text{nonrel}} = (\hat{\mathbf{p}}^2/2m) + q\Phi$, [see Eq. (4.3.71)].

Dirac equation: Historically the fact that the ρ of the KGE was not positive definite and the existence of negative energy solutions drove Dirac to seek alternative forms of RQM. It was the occurrence of $\hat{H}^2 = -\hbar^2\partial^2/\partial t^2$ in the KGE that led to the latter, so he sought an equation *linear* in $\hat{H} = i\hbar\partial/\partial t$. He proposed the simple form known as *the Dirac equation*,

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = \left[-i\hbar c\,\alpha\cdot\nabla + \beta mc^2\right]\psi\,.$$

To satisfy $E^2 = \mathbf{p}^2c^2 + m^2c^4$ requires

$$-\hbar^2\partial^2\psi/\partial t^2 = \hat{H}^2\psi = [\hat{\mathbf{p}}^2c^2 + m^2c^4]\psi = [-\hbar^2c^2\nabla^2 + m^2c^4]\psi = 0$$

and so It then follows that α and β must obey the anticommutation relations of a Clifford algebra,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}I \quad , \quad \{\alpha_i, \beta\} = 0 \quad , \quad \beta^2 = I \quad \text{for} \quad i,j = 1,2,3\,,$$

where the anticommutator is $\{A, B\} \equiv AB + BA$ (not to be confused with the Poisson bracket!). we also have $\text{tr}\alpha_i = -\text{tr}(\beta\alpha_i\beta) = -\text{tr}(\beta^2\alpha_i) = -\text{tr}\alpha_i = 0$ and $\text{tr}\beta = -\text{tr}(\alpha_i\beta\alpha_i) = -\text{tr}(\alpha_i^2\beta) = -\text{tr}\beta = 0$. So α and β must be matrices and the lowest possible dimensional representation is as 4×4 matrices.

Relativistic Quantum Mechanics (RQM)

(See Chapter 4, Secs 4.3 & 4.4)

The Dirac representation is $\alpha_i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$,

where the σ^i are the usual Pauli spin matrices and $\psi(x)$ is called a *Dirac spinor wavefunction*. It is easily checked that the Dirac equation leads to the conserved current

$$j^\mu = (j^0, \mathbf{j}) = (c\rho, \mathbf{j}) \equiv (\psi^\dagger\psi, c\psi^\dagger\alpha\psi) \quad \text{where} \quad \partial_\mu j^\mu = 0 .$$

Note that ρ is positive definite here, $\rho = \psi^\dagger\psi > 0$.

Lorentz covariance and the Dirac equation: We should express the Dirac equation in terms of $x^\mu = (ct, \mathbf{x})$and $\partial_\mu = (\partial/c\partial t, \nabla)$. In order to achieve this we define $\gamma^0 \equiv \beta$ and $\gamma^i \equiv \gamma^0\alpha_i = \beta\alpha_i$

which gives the Dirac representation of the γ-matrices $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$.

The Dirac equation then takes the covariant-looking form

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) = (i\hbar\partial\!\!\!/ - mc)\psi(x) = 0 ,$$

where we have introduced the *Feynman "slash" notation* $A\!\!\!/ \equiv \gamma^\mu A_\mu = \gamma_\mu A^\mu$. In order for this equation to be Lorentz covariant we require a 4×4 matrix representation $S(\Lambda)$ of the the restricted Lorentz transformations $\Lambda^\mu{}_\nu$ such that

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu \quad \text{and} \quad \psi(x) \longrightarrow \psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x) .$$

With this we find that the Dirac equation actually becomes covariant as required,

$$0 = (i\hbar\gamma^\mu\partial'_\mu - mc)\psi'(x') = (i\hbar\gamma^\mu\partial_\mu - mc)\psi(x) .$$ [See Eqs. (4.4.63)-(4.4.65)]

Relativistic Quantum Mechanics (RQM)

(See Chapter 4, Secs 4.3 & 4.4)

The form of $S(\Lambda)$ can be derived [see Eqs. (4.4.67)-(4.4.75)] and is given by

$$S(\Lambda) = \exp\left(-\tfrac{i}{2}\omega_{\rho\sigma}\Sigma^{\rho\sigma}_{\text{Dirac}}/\hbar\right) = \exp\left(-\tfrac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right)$$

where we have defined $\sigma^{\rho\sigma} \equiv (i/2)\left[\gamma^\rho, \gamma^\sigma\right]$ and $\Sigma^{\rho\sigma}_{\text{Dirac}} \equiv \frac{1}{2}\hbar\sigma^{\rho\sigma}$.

The relation to the $\Sigma^{\mu\nu}$ discussed earlier in relativistic classical field theory is $\Sigma^{\rho\sigma}_{\text{Dirac}} \equiv i\Sigma^{\rho\sigma}$.

We define the *Dirac adjoint spinor* as $\bar\psi(x) \equiv \psi(x)^\dagger\gamma^0$. It follows that for a restricted Lorentz transformation $\bar\psi(x) \to \bar\psi(x)S(\Lambda)^{-1}$ since

$$\bar\psi(x) \to \bar\psi'(x') = \psi'(x)^\dagger\gamma^0 = \psi(x)^\dagger S(\Lambda)^\dagger\gamma^0 = \bar\psi\gamma^0 S(\Lambda)^\dagger\gamma^0 = \bar\psi(x)S(\Lambda)^{-1}.$$

It then follows for example that $\bar\psi(x)\psi(x)$ *is a Lorentz scalar* and $\bar\psi(x)\gamma^\mu\psi(x)$ *is a Lorentz four-vector*.

The plane wave solutions to the Dirac equation are $\psi(x) = e^{-ip\cdot x/\hbar}u^s(p) = e^{-ip\cdot x/\hbar}S(\Lambda)u^s(0)$ for particle solutions and $\psi(x) = e^{+ip\cdot x/\hbar}v^s(p) = e^{+ip\cdot x/\hbar}S(\Lambda)v^s(0)$ for antiparticle solutions, where we have

$$u^s(p) = S(\Lambda)u^s(0) = \frac{\not p + mc}{\sqrt{(p^0+mc)}}\begin{pmatrix}\phi_s\\0\end{pmatrix}, \qquad u^s(0) = \sqrt{2mc}\begin{pmatrix}\phi_s\\0\end{pmatrix},$$

$$v^s(p) = S(\Lambda)v^s(0) = \frac{-\not p + mc}{\sqrt{(p^0+mc)}}\begin{pmatrix}0\\\chi_{-s}\end{pmatrix}, \qquad v^s(0) = \sqrt{2mc}\begin{pmatrix}0\\\chi_{-s}\end{pmatrix}.$$

Note the $u^\pm(p)$ and $v^\pm(p)$ correspond to $s = \pm\frac{1}{2}$ particles and antiparticles respectively and where ϕ_s and χ_s are two component spinors defined such that $\phi_+ = \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\phi_- = \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$.

Relativistic Quantum Mechanics (RQM)

(See Chapter 4, Secs 4.3 & 4.4)

With $p^\mu = (p^0, \mathbf{p}) = (\gamma mc, \gamma m\mathbf{v})$ we have $p^2 = m^2c^2$ and so

$$\not{p}^2 = \not{p}\not{p} = p_\mu p_\nu \gamma^\mu \gamma^\nu = \frac{1}{2} p_\mu p_\nu \{\gamma^\mu, \gamma^\nu\} = p_\mu p_\nu g^{\mu\nu} = p^2 = m^2c^2$$

So we have

$$(\not{p} - mc)(\not{p} + mc) = (\not{p} + mc)(\not{p} - mc) = \not{p}2 - m^2c^2 = p^2 - mc^2 = 0\,,$$
$$(\not{p} + mc)^2 = p^2 + 2mc\ \not{p} + m^2c^2 = 2mc(\not{p} + mc)$$
$$(-\not{p} + mc)^2 = p^2 - 2mc\ \not{p} + m^2c^2 = 2mc(-\not{p} + mc)\,.$$

It then follows from the forms given for $u^s(p)$ and $v^s(p)$ that

$$(\not{p} - mc)u^s(p) = 0 \quad \text{and} \quad (\not{p} + mc)v^s(p) = 0\,.$$

We verify that $\psi(x) = e^{-ip\cdot x/\hbar}u^s(p)$ and $\psi(x) = e^{+ip\cdot x/\hbar}v^s(p)$ are solutions of the Dirac equation, since

$$(i\hbar\not{\partial} - mc)e^{-ip\cdot x/\hbar}u^s(p) = (\not{p} - mc)e^{-ip\cdot x/\hbar}u^s(p) = 0 \quad \text{and}$$
$$(i\hbar\not{\partial} - mc)e^{+ip\cdot x/\hbar}v^s(p) = -(\not{p} + mc)e^{+ip\cdot x/\hbar}v^s(p) = 0\,.$$

Using the above results we can show for all $s, s' = \pm\frac{1}{2}$ that

$$\bar{u}^s(p)u^{s'}(p) = 2mc\delta^{ss'} \quad \text{and} \quad \bar{v}^s(p)v^{s'}(p) = -2mc\delta^{ss'}\,,$$
$$\bar{v}^s(p)u^{s'}(p) = \bar{u}^s(p)v^{s'}(p) = 0 \quad \text{and} \quad v^{s\dagger}(p)u^{s'}(-p) = u^{s\dagger}(p)v^{s'}(-p) = 0\,,$$
$$\bar{u}^s(p)\not{p} = mc\,\bar{u}^s(p) \quad \text{and} \quad \bar{v}^s(p)\not{p} = -mc\,\bar{v}^s(p)\,,$$
$$u^s(p)^\dagger u^{s'}(p) = v^s(p)^\dagger v^{s'}(p) = (2E/c)\delta^{ss'}\,,$$
$$p^\mu = \frac{1}{2}\bar{u}^s(p)\gamma^\mu u^s(p) = \frac{1}{2}\bar{v}^s(p)\gamma^\mu v^s(p)\,.$$

Relativistic Quantum Mechanics (RQM)

(See Chapter 4, Secs 4.3 & 4.4 and Appendix A, Sec. A.3)

We can form fermion bilinear that transform under Lorentz transformations as:

$$\begin{aligned}
\text{scalar:}\quad & \bar\psi'(x')\psi'(x') = \bar\psi(x)\psi(x)\,,\\
\text{vector:}\quad & \bar\psi'(x')\gamma^\mu\psi'(x') = \Lambda^\mu{}_\nu\bar\psi(x)\gamma^\nu\psi(x)\,,\\
\text{second-rank tensor:}\quad & \bar\psi'(x')\sigma^{\mu\nu}\psi'(x') = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar\psi(x)\sigma^{\rho\sigma}\psi(x)\,,\\
\text{pseudovector/axial vector:}\quad & \bar\psi'(x')\gamma^\mu\gamma^5\psi'(x') = \det(\Lambda)\Lambda^\mu{}_\nu\bar\psi(x)\gamma^\nu\gamma^5\psi(x)\,,\\
\text{pseudoscalar:}\quad & \bar\psi'(x')i\gamma^5\psi'(x') = \det(\Lambda)\bar\psi(x)i\gamma^5\psi(x)\,,
\end{aligned}$$

The Dirac γ-matrices satisfy a number of important identities, some of which are:

$$\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \;\Rightarrow\; (\gamma^0)^2 = -(\gamma^i)^2 = I\,,$$

$$\gamma^0\gamma^i = -\gamma^i\gamma^0\,,\quad \gamma^0\gamma^i\gamma^0 = -\gamma^i\,,\quad \gamma^i\gamma^0\gamma^i = \gamma^0 \;\Rightarrow\; \mathrm{tr}\gamma^i = \mathrm{tr}\gamma^0 = 0\,,$$

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,,\quad \sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]\,,$$

$$\{\gamma^\mu,\gamma^5\} = 0\,,\quad (\gamma^5)^2 = I\,,\quad \gamma^\mu\gamma^\nu = \tfrac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu] = g^{\mu\nu} - i\sigma^{\mu\nu}\,,$$

$$\not{p}\,\not{q} = p\cdot q - i\sigma_{\mu\nu}p^\mu q^\nu\,,\quad \left[\gamma^5,\sigma^{\mu\nu}\right] = 0 \quad\text{and}\quad \gamma^5\sigma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\,.$$

For a full list of Dirac γ-matrix representations and identities, including trace and contraction identities, see the Appendix, Sec. A.3.