Foundations of Quantum Field Theory

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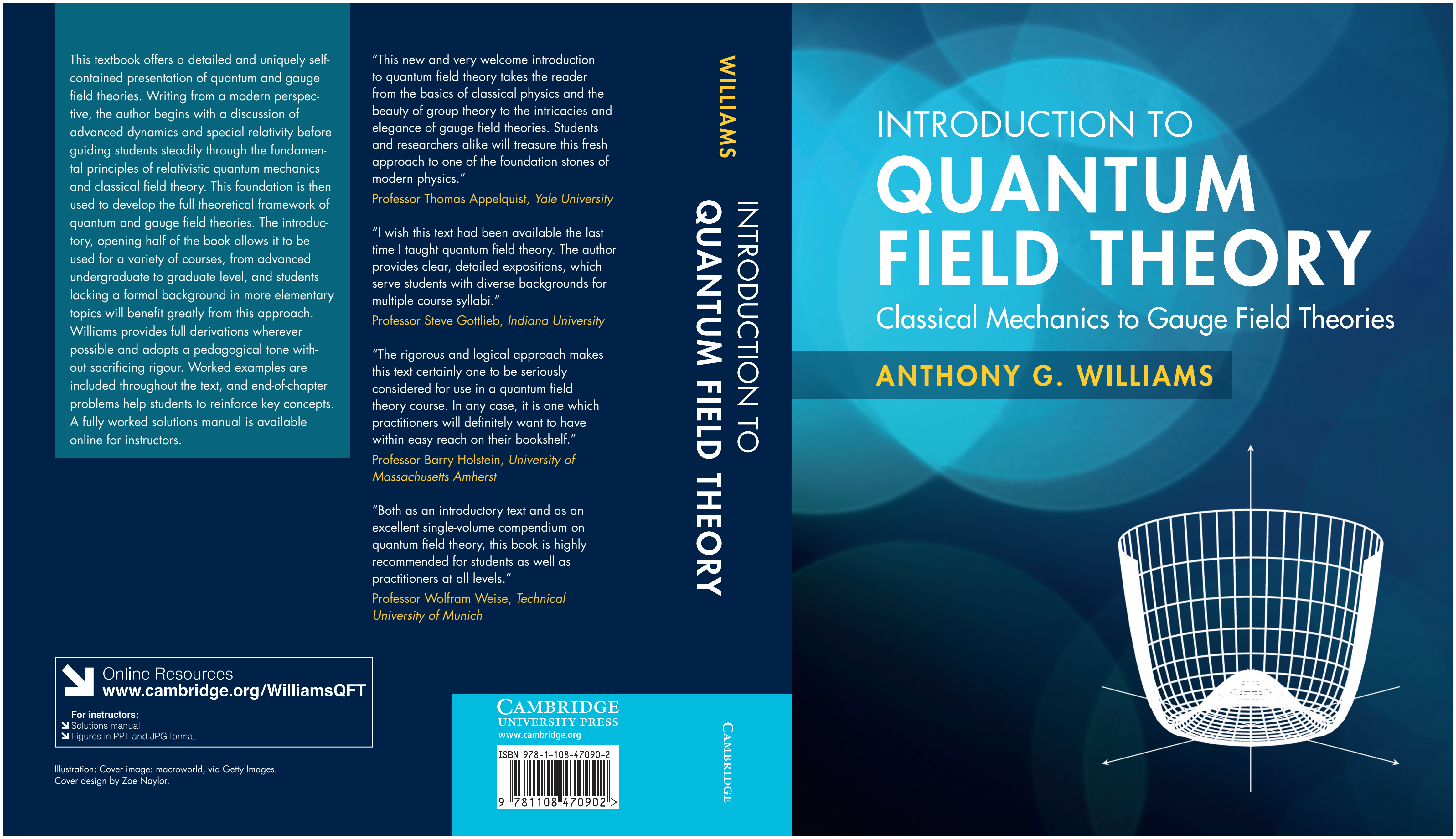
Primary reference

- Hard cover printed version from Cambridge University Press;
- Amazon Kindle digital version has identical appearance and page numbering as the printed version - other digital versions also available with different appearance and variable page numbering.



References

The primary source for these lectures is my QFT textbook:

(i) *Introduction to quantum Field Theory: Classical Mechanics to Gauge Field Theories,* Anthony G. Williams (Cambridge University Press, 2022)

Note: *Any references to sections and/or equation numbers in these lectures refer to this textbook.*

Other recommended textbooks using similar notation and conventions:

(ii) *Introduction to Quantum Field Theory*, Michael E. Peskin and Daniel V. Schroeder, (CRC Press, 2019)

(iii) *Quantum Field Theory and the Standard Model*, Matthew D. Schwartz, (Cambridge University Press, 2022)

Other useful references include: Schweber (1961), Bjorken and Drell (1964 and 1965), Roman (1969), Nash (1978), Itzykson and Zuber (1980), Cheng and Li (1984), Mandl and Shaw (1984), Ryder (1986), Brown (1992), Bailin and Love (1993), Sterman (1993), Weinberg (1995 and 1996), Greiner and Reinhardt (1996), Pokorski (2000), Srednicki (2007), Zee (2010), Aitchison and Hey (2013) and many others.

Lecture 1: Lightening Review of Assumed Knowledge

Elements of group theory, special relativity, classical mechanics, and quantum mechanics

The following is far too much for one lecture!!! Review and refer back to this material as needed. This lecture establishes a common platform for students of varying backgrounds so that the remaining lectures can be built on it.

Group Theory & Lie Groups: Cheat Sheet

See Vanessa Robbins lectures for careful discussion and details and the Appendix, Sec. A.7

Definition of a group:

A set of elements $\{g_1, g_2, g_3, \cdots\}$ forms a *group*, G, when a binary group operation (or group multiplication), $g_i g_j \equiv g_i \circ g_j$, has the properties:

(i) *Closure:* If $g_i, g_j \in G$ then $g_i g_j \in G$, i.e., $g_i g_j = g_k$ for some $g_k \in G$.

(ii) *Associativity:* $g_i(g_j g_k) = (g_i g_j) g_k$ for all $g_i, g_j, g_k \in G$.

(iii) *Identity:* There exists some e (or we will often write I) $\in G$, called the identity, such that $e g_i = g_i e = g_i$ for every $g_i \in G$.

(iv) *Inverse:* For every $g_i \in G$ there exists some $g_i^{-1} \in G$, called the inverse of g_i, such that $g_i g_i^{-1} = g_i^{-1} g_i = e$.

Lie groups:

A continuous group with $g_i \to g(\vec{\omega}) \in G$ is a *Lie group* if it is infinitely differentiable with respect to its real parameters $\vec{\omega}$. So Lie groups are also real smooth manifolds. This is the definition of a *real Lie group* and includes, e.g., $U(n)$, $SU(n)$, $O(n)$ and $SO^+(1,3)$. There are also *complex Lie groups* which are complex analytic in their complex group parameters $\vec{\omega}$.

Important examples of complex Lie groups are $GL(n, \mathbb{C})$ (invertible complex $n \times n$ matrices) and $SL(n, \mathbb{C})$ (complex $n \times n$ matrices with unit determinant).

Consider a real Lie group with elements $g(\vec{\omega})$. Sufficiently close to the identity we can define

$$g(\vec{\omega}) = I + i\vec{\omega} \cdot \vec{T} + \mathcal{O}(\omega^2) = I + i\omega^a T^a + \mathcal{O}(\omega^2) \quad , \text{ where we can define } \left.\frac{\partial g}{\partial \omega^a}\right|_{\vec{\omega}=\vec{0}} \equiv iT^a$$

where the T^a are called the *generators* of the Lie group G.

Group Theory & Lie Groups: Cheat Sheet

Provided $g(\vec{\omega})$ is sufficiently close to the identity then we can write

$$g(\vec{\omega}) = \lim_{N\to\infty} g(\vec{\omega}/N)^N = \lim_{N\to\infty} [I + i(1/N)\vec{\omega}\cdot\vec{T}]^N = e^{i\vec{\omega}\cdot\vec{T}}.$$

A real Lie group G will not be homeomorphic (a topology preserving mapping) to $\mathbb{R}^n$, i.e., the above equation cannot hold for all elements $g \in G$ of the Lie group.

Consider the product of two group elements near the identity. Closure means that for some $\epsilon_3\vec{\omega}_3$,

$$g(\epsilon_1\hat{\omega}_1)g(\epsilon_2\omega_2) = \exp\{i(\epsilon_1\hat{\omega}_1 + \epsilon_2\hat{\omega}_2)\cdot\vec{T} - \tfrac{1}{2}\epsilon_1\epsilon_2\omega_1^a\omega_2^b[T^a, T^b] + \cdots\} = \exp\{i\epsilon_3\hat{\omega}_3\cdot\vec{T}\} = g(\epsilon_3\hat{\omega}_3)$$

by the Baker-Campbell-Hausdorff theorem and so it must be true that we can write

$[T^a, T^b] = if^{abc}T^c$ (defines the *Lie algebra* of the Lie Group) **Note:** *We use summation convention!*

for some constants $f^{abc} \in \mathbb{R}$. We call the f^{abc} the *structure constants* of the Lie algebra. For some Lie group G we write its Lie algebra using lower case gothic font as $\mathfrak{g}$. Any element of the real Lie algebra $\mathfrak{g}$ for a real Lie group G is a real linear combination the group generators T^a, i.e., $c^aT^a \in \mathfrak{g}$ for all $c^a \in \mathbb{R}$.

The Lie algebra is the tangent space to the Lie Group at the identity. Different Lie Groups can have the same Lie algebra, e.g., the Lie groups $O(3)$ and $SU(2)$ have the same Lie algebra, $\mathfrak{su}(2) = \mathfrak{o}(3)$, defined by $[(J^i/\hbar), (J^j/\hbar)] = i\epsilon^{ijk}(J^k/\hbar)$ with generators $(J^i/\hbar)$ and structure constants $f^{ijk} = i\epsilon^{ijk}$. It can be shown that $SU(2)$ is a *double cover* of $SO(3)$ in that every element of $SO(3)$ has two corresponding elements of $SU(2)$. We often work in natural units where $\hbar = c = 1$ so that $(J^i/\hbar) \to J^i$.

The (quadratic) Casimir invariants of a Lie group G are quadratic combinations of the generators that commute with all elements of the Lie algebra $\mathfrak{g}$, e.g., the only such Casimir invariant of $SU(2)$ is $\mathbf{J}^2$, since it commutes with every linear combination of the J^i or equivalently since $[\mathbf{J}^2, J^i] = 0$. Obviously any polynomial of $\mathbf{J}^2$ also commutes with all of the generators.

Group Theory & Lie Groups: Cheat Sheet

Representations of group: A *representation* of a group G is a mapping $G \to GL(V)$, where $GL(V)$ is the general linear group on the vector space V, e.g., $V = \mathbb{C}^n$. Note that $GL(\mathbb{C}^n) \equiv GL(n, \mathbb{C})$ is the set of $n \times n$ invertible complex matrices. Different matrix realizations of the representation are related by changes of basis of V, i.e., by similarity transformations. In a specific realization in terms of matrices there is a matrix $D(g)$ for every $g \in G$ and the group operation corresponds to matrix multiplication.

If every $D(g)$ can be brought into block-diagonal form by some constant similarity transformation S then we have a *completely reducible representation*, i.e., if there is an S such that $SD(g)S^{-1}$ has a block diagonal form

$$SD(g)S^{-1} = \mathrm{diag}[D_1(g), D_2(g), D_3(g), \cdots],$$

where each block $D_j(g)$ is also an irreducible representation of G.

We can also have *infinite dimensional representations* of groups in terms of differential operators, such as the unitary representation of the translation group in quantum mechanics, $\exp(-i\mathbf{a} \cdot \mathbf{P})$ with the total three-momentum operator $\mathbf{P} = -\mathbf{i}\hbar\nabla$ in coordinate-space representation.

If all of the group elements g commute with each other we say that the group is *abelian*, otherwise we say that it is a *nonabelian* group. All structure constants of an abelian Lie group must therefore vanish, since the Lie group elements g only commute if all of the generators T^a do.

Lorentz and Poincaré Invariance

(See Chapter 1, Sec 1.2)

The postulates of special relativity are:
(i) the laws of physics are the same in all inertial frames; and
(ii) the speed of light is constant and is the same in all inertial frames.

Let E_1 and E_2 be any two spacetime events labeled by inertial observer $\mathcal{O}$ as $x_1^\mu = (ct_1, \mathbf{x}_1)$ and $x_2^\mu = (ct_2, \mathbf{x}_2)$ respectively. The *spacetime displacement* of these two events in the frame of $\mathcal{O}$ is

$$z^\mu \equiv \Delta x^\mu = (x_2^\mu - x_1^\mu) = (c(t_2 - t_1), \mathbf{x}_2 - \mathbf{x}_1) = (c\Delta t, \Delta \mathbf{x}) .$$

The Minkowski-space *metric tensor*, $g_{\mu\nu}$, is defined by

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = +1 \qquad \text{and} \qquad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu ,$$

which we can also write as $g_{\mu\nu} \equiv \mathrm{diag}(1, -1, -1, -1)$. We can then further define

$$x^2 \equiv x^\mu g_{\mu\nu} x^\nu = (x^0)^2 - \mathbf{x}^2 = (ct)^2 - \mathbf{x}^2 ,$$

where we use the *Einstein summation convention* that repeated spacetime indices are understood to be summed over. In general relativity $g_{\mu\nu} \to g(x)_{\mu\nu}$.

The postulates immediately lead to the result that changing from one inertial frame to another implies

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \text{or in matrix form} \qquad x \to x' = \Lambda x ,$$

where $\Lambda^\mu{}_\nu$ or more briefly Λ is a *Lorentz transformation* consisting of 16 real constants and where invariance of the speed of light gives the result that

$$\Lambda^T g \Lambda = g \qquad \text{or equivalently} \qquad \Lambda^\sigma{}_\mu g_{\sigma\tau} \Lambda^\tau{}_\nu = g_{\mu\nu} .$$

The set of all such Λ form a group called the *Lorentz group*, which is denoted as $O(1,3)$ in analogy with the rotation group of 3×3 matrices $O(3)$ that satisfy $O^T O = I$ or equivalently $O^T(-I)O = (-I)$.

Lorentz & Poincaré Invariance

(See Chapter 1, Sec 1.2)

Table 1.1. Classes of the Lorentz transformations

Lorentz class	$\det\Lambda$	$\Lambda^0{}_0$	Transformation
(i) Rotations and Boosts	$+1$	≥ 1	$(\Lambda_r)^\mu{}_\nu \in SO^+(1,3)$
(ii) Parity inverting	-1	≥ 1	$(P\Lambda_r)^\mu{}_\nu = P^\mu{}_\sigma(\Lambda_r)^\sigma{}_\nu;\ P^\mu{}_\nu \equiv g^{\mu\nu}$
(iii) Time reversing	-1	≤ -1	$(T\Lambda_r)^\mu{}_\nu = T^\mu{}_\sigma(\Lambda_r)^\sigma{}_\nu;\ T^\mu{}_\nu \equiv -g^{\mu\nu}$
(iv) Spacetime invert	$+1$	≤ -1	$(PT\Lambda_r)^\mu{}_\nu = (PT)^\mu{}_\sigma(\Lambda_r)^\sigma{}_\nu;\ (PT)^\mu{}_\nu \equiv -\delta^\mu{}_\nu$

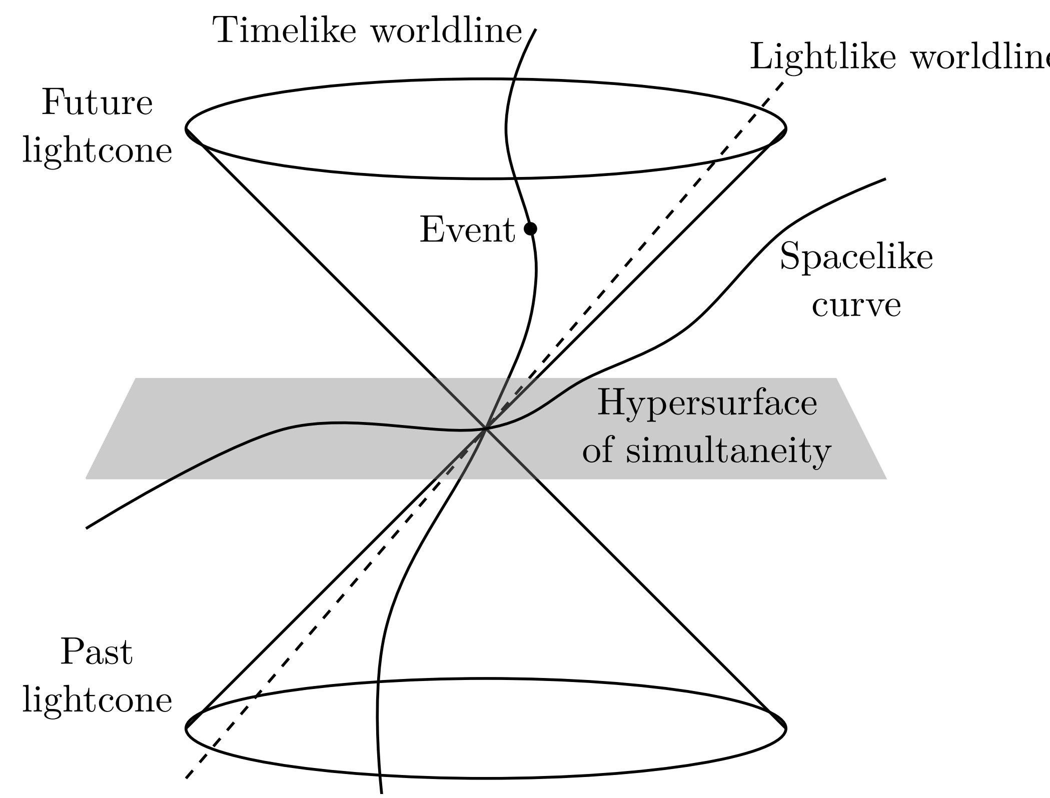


Table 1.2. Classification of spacetime functions based on Lorentz transformation properties in the passive view, ($\det(\Lambda) = \pm 1$)

$s(x)$	$\to$	$s'(x') = s(x)$	Scalar
$p(x)$	$\to$	$p'(x') = \det(\Lambda)p(x)$	Pseudoscalar
$v^\mu(x)$	$\to$	$v'^\mu(x') = \Lambda^\mu{}_\nu v^\nu(x)$	Vector
$a^\mu(x)$	$\to$	$a'^\mu(x') = \det(\Lambda)\Lambda^\mu{}_\nu a^\nu(x)$	Pseudovector (or axial vector)
$t^{\mu\nu}(x)$	$\to$	$t'^{\mu\nu}(x') = \Lambda^\mu{}_\sigma\Lambda^\nu{}_\tau t^{\sigma\tau}(x)$	Second-rank tensor

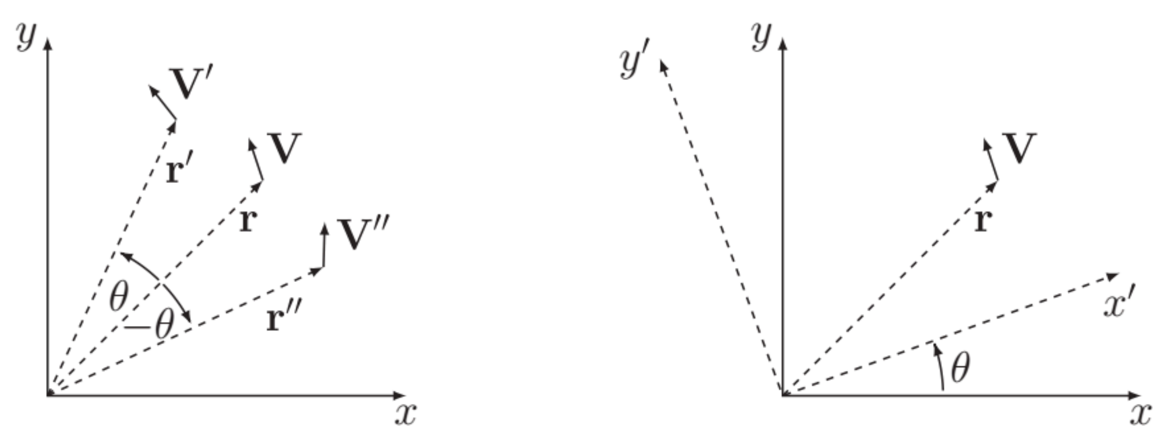


Figure 1.3 The left-hand figure illustrates active rotations by angles θ and $-\theta$, where the vector $\mathbf{V}(\mathbf{r})$ represents the physical system. The right-hand figure illustrates a passive rotation by angle θ of the coordinate system of the observer, $\mathcal{O} \to \mathcal{O}'$.

Lorentz and Poincaré Invariance

(See Chapter 1)

The rotations and boosts form the group of *restricted Lorentz transformations* denoted as $SO^+(1,3)$, which have $\det \Lambda = 1$ (`special' $\Longrightarrow S$) and $\Lambda^0{}_0 \geq 1$ ($\Longrightarrow$ $+$). This is the continuous subgroup of the Lorentz group $O(1,3)$.

For any of the Λ sufficiently close to the identity we can write for some small arbitrary $\omega_{\mu\nu}$ and fixed linearly independent 4×4 real antisymmetric matrices $M^{\mu\nu}$

$$\Lambda^\alpha{}_\beta = \left[\exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)\right]^\alpha{}_\beta ,$$

where the $M^{\mu\nu}$ are recognized as the generators of the Lie group $SO^+(1,3)$. So the restricted Lorentz group $SO^+(1,3)$ is a Lie group. There are only 6 independent $M^{\mu\nu}$ since there are at most 6 linearly independent 4×4 real antisymmetric matrices.

- Rotations make up 3 (rotations about x, y and z) with $J^i \equiv \frac{1}{2}\epsilon^{ijk}M^{jk}$; and
- Lorentz boosts make up 3 (Lorentz boosts in x, y and z directions) with $K^i = M^{0i}$.

It can be shown from the defining relationship $\Lambda^T g \Lambda = g$ that the Lorentz Lie algebra must have the form

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}\right) .$$

It is conventional to write the Lie algebra for the restricted Lorentz transformations in this form rather than in terms of its structure constants. The 4×4 representation is the *defining representation* of the Lorentz group. There are two Casimir invariants of the Lorentz group and these are $\mathbf{J}^2$ and $\mathbf{K}^2$.

The *Poincaré group* is the group made up of spacetime translations and Lorentz transformations.

Unitary Representation of the Poincaré Group

(See Chapter 1)

Recall that in quantum mechanics for a spinless particle the three-momentum and angular momentum operators in coordinate space representation are respectively

$\mathbf{P} \equiv -i\hbar\nabla$ and $\mathbf{J} = \mathbf{L} \equiv \mathbf{x} \times \mathbf{P}$ or equivalently $J^i = \epsilon^{ijk} x^j P^k = \frac{1}{2}\epsilon^{ijk} M^{jk}$,

where we have defined here

$$M^{ij} = x^i P^j - x^j P^i \,.$$

Generalizing these three-vector quantities to four-vectors leads to the Hermitian quantities

$P^\mu \equiv (H/c, \mathbf{P}) \equiv i\hbar\partial^\mu$ and $M^{\mu\nu} \equiv x^\mu P^\nu - x^\nu P^\mu$.

The P^μ are the generators of the spacetime translations and commute with each other (they form an abelian group) and the $M^{\mu\nu}$ are the generators of the (nonabelian) Lorentz group since they can be shown to satisfy the Lorentz Lie algebra. It is also readily shown that $\left[P^\mu, M^{\rho\sigma}\right] = i\hbar(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho)$.

Since the generators P^μ and $M^{\mu\nu}$ are Hermitian then the Lie group elements are *unitary* and so we have a *unitary representation*.

We have arrived at the *Poincaré Lie algebra* consisting of the 4+6=10 generators, P^μ and $M^{\mu\nu}$,

$$\left[P^\mu, P^\nu\right] = 0\,, \qquad \left[P^\mu, M^{\rho\sigma}\right] = i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho),$$

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}\right).$$

Generalizing to include spin we have $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and $M^{\mu\nu} = L^{\mu\nu} + \Sigma^{\mu\nu}$ with $L^{\mu\nu}$ generating orbital rotations and $\Sigma^{\mu\nu}$ generating rotations of intrinsic spin. The angular momentum $\mathbf{J}$ is a three-vector and not a convenient quantity in a relativistic framework. Instead we use th Pauli-Lubanski pseudovector

$$W_\mu \equiv -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \,.$$

Unitary Representation of the Poincaré Group

(See Chapter 1)

The Poincaré group has only two Casimir invariants,

$$P^2 = m^2c^2 \quad \text{and} \quad W^2 = -m^2 s(s+1)\ ,$$

where s is the spin of the particle in its rest frame and m is the particle rest mass. A massive particle has $2s+1$ possible spin states in the usual way, $m_s = s, s-1, \ldots, -s+1, -s$, e.g., for spin-half particles $m_s = \pm\frac{1}{2}$. It can also be shown that for massless particles only two helicity states are possible, $\lambda = \pm s$, e.g., for photons the only allowed spin states are $\lambda = \pm 1$.

Summary: Particles are categorized according to the representation of the Poincaré group that they transform under, which is specified by their Casimir invariants and hence by their mass m and their spin s. Allowed values for s are the non-negative half-integers, $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Classical mechanics: Lagrangian formulation

(See Chapter 2, Secs 2.1 and 2.3)

Lagrangians: For a classical system with holonomic constraints and the resulting generalized coordinates $q_1, \ldots, q_N$ and where the potential for the system is monogenic (includes conservative potentials and electromagnetic interactions), the equations of motion are given by the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{and} \qquad \dot{q}_i = \frac{dq_i}{dt} \qquad \text{for} \qquad i = 1, \cdots N,$$

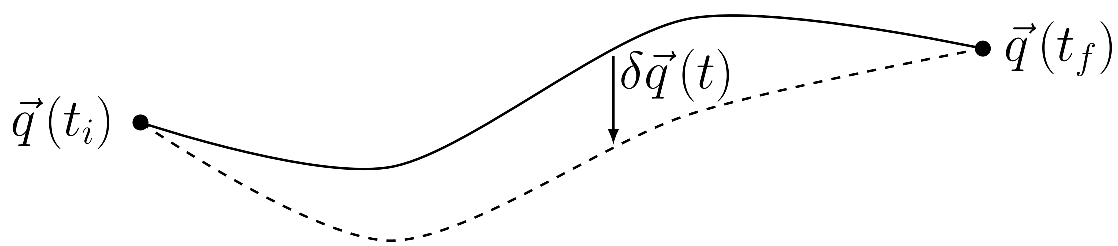
where $L \equiv T - V$ is the Lagrangian for the system, T is the kinetic energy and V is the potential.

The action associated with the system is a functional of the path in coordinate space between initial time t_i and final time t_f and is defined as

$$S[\vec{q}\,] \equiv \int_{t_i}^{t_f} dt\, L(\vec{q}, \dot{\vec{q}}, t)\,.$$

Hamilton's Principle is that the equations of motion correspond to a stationary point of the action under variations of the classical path with end-points fixed, which follows since we can show that

$$0 = \frac{\delta S[\vec{q}\,]}{\delta q_j(t)} = \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \quad \text{for} \quad j = 1, 2, \ldots, N.$$



Normal modes: For sufficiently small oscillations around a stable equilibrium the leading terms are the part of T quadratic in velocities $\dot{q}_j$ and the part of V quadratic in the coordinates q_j. The resulting approximate Lagrangian L then has the form of N coupled harmonic oscillators, which after diagonalization reduces to N independent harmonic oscillators called *normal modes*. Any sufficiently small classical motion of the system can be expressed as a linear superposition of the normal modes.

Classical mechanics: Hamiltonian formulation

(See Chapter 2, Sec. 2.4)

Hamiltonians: The Lagrangian *Hessian matrix* M_L is defined by

$$(M_L)_{ij} \equiv \partial^2 L/\partial\dot{q}_i\partial\dot{q}_j \;.$$

M_L is real and symmetric and so can be diagonalized by an orthogonal transformation with its real eigenvalues down the diagonal.

We define the *generalized (i.e., canonical) momenta* as

$$p_i(t) \equiv \partial L/\partial\dot{q}_i \quad \text{or equivalently} \quad \vec{p} \equiv \partial L/\partial\dot{\vec{q}} \;.$$

Provided that M_L is a positive definite matrix (only positive eigenvalues $\Longrightarrow$ concave up) we can define the Hamiltonian for the system as a Legendre transform of the Lagrangian with respect to the generalized velocities as

$$H(\vec{q},\vec{p},t) \equiv \left(\sum_{i=1}^{N} p_i\dot{q}_i\right) - L(\vec{q},\dot{\vec{q}}(\vec{p}),t) = \vec{p}\cdot\dot{\vec{q}} - L \;.$$

It is not difficult to show that the Euler-Lagrange equations are equivalent to Hamilton's Equations, which are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \frac{dH}{dt} = -\frac{\partial L}{\partial t} \;.$$

So Hamilton's equations, Hamilton's Principle and the Euler-Lagrange equations are *three equivalent ways* of describing the classical motion of such systems.

If M_L has a mixture of positive and negative eigenvalues the energy is typically not bounded below and such systems are unphysical and so not relevant. Some important physical systems (e.g., gauge theories and theories with fermions) can have a mixture of positive and zero eigenvalues. Such systems are called *singular systems* and need to be treated carefully using *constrained Hamiltonian dynamics*, *the Dirac-Bergmann algorithm* and the *Dirac bracket*. We cannot discuss these subtleties here but they are treated in detail for classical systems in Sec. 2.9.

Classical mechanics: Hamiltonian formulation

(See Chapter 2, Sec. 2.4.2)

Poisson brackets: The *Poisson bracket* formulation of classical mechanics underlies the deep connection between classical and quantum physics as we will soon show. It is the basis of Dirac's canonical quantization of classical systems. We refer to the space of points $(\vec{q},\vec{p})$ as *phase space* and any doubly differentiable function of phase space is a *dynamical variable.* Dynamical variables may have an explicit time dependence and so can be written as $F(\vec{q}(t),\vec{p}(t),t)$. The Poisson bracket of two dynamical variables F and G is defined as

$$\{F,G\} \equiv \sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right).$$

If A, B and C are dynamical variables and if a and b are real constants, then it follows that

(i) Closure: $\{A,B\}$is also a function on phase space;

(ii) Antisymmetry: $\{A,B\} = -\{B,A\}$;

(iii) Bilinearity: $\{aA+bB,C\} = a\{A,C\}+b\{B,C\}$;

(iv) Product rule: $\{A,BC\} = \{A,B\}C+B\{A,C\}$;

(v) Jacobi identity: $\big\{A,\{B,C\}\big\}+\big\{B,\{C,A\}\big\}+\big\{C,\{A,B\}\big\}=0.$

Note: These are identical to the properties of the commutator $[\cdots,\cdots]$.

It is easily seen that

$$\frac{dF}{dt}=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{dq_j}{dt}+\frac{\partial F}{\partial p_j}\frac{dp_j}{dt}\right)+\frac{\partial F}{\partial t}=\sum_{j=1}^{N}\left(\frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j}-\frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j}\right)+\frac{\partial F}{\partial t}=\{F,H\}+\frac{\partial F}{\partial t},$$

which is the analog of Ehrenfest's theorem in quantum mechanics $\dfrac{d\hat{F}}{dt}=\dfrac{1}{i\hbar}[\hat{F},\hat{H}]+\dfrac{\partial\hat{F}}{\partial t}$.

Relation between Quantum Mechanics and Classical Mechanics

(See Chapter 2, Sec. 2.5)

For any Heisenberg picture operator, $\hat{F}(t) \equiv F(\hat{\vec{q}}(t), \hat{\vec{p}}(t), t)$, the operator form of Ehrenfest's theorem is

$$\frac{d\hat{F}}{dt} = \frac{1}{i\hbar}[\hat{F}, \hat{H}] + \frac{\partial \hat{F}}{\partial t},$$

where $\hat{H}(t) \equiv H(\hat{\vec{q}}(t), \hat{\vec{p}}(t), t)$ is the Heisenberg-picture quantum Hamiltonian operator. The canonical commutation relations are

$$\left[\hat{q}^i(t), \hat{p}^j(t)\right] = i\hbar\delta^{ij}, \ \left[\hat{q}^i(t), \hat{q}^j(t)\right] = \left[\hat{p}^i(t), \hat{p}^j(t)\right] = 0\,.$$

It is is easily shown by induction that we can define operator differentiation using

$$[\hat{q}_i, \hat{p}_j^n] = i\hbar\delta_{ij}\left(n\hat{p}_i^{n-1}\right) \equiv i\hbar\frac{\partial \hat{p}_j^n}{\partial \hat{p}_i}, \quad [\hat{p}_i, \hat{q}_j^n] = -\,i\hbar\delta_{ij}\left(n\hat{q}_i^{n-1}\right) \equiv -\,i\hbar\frac{\partial \hat{q}_j^n}{\partial \hat{q}_i},$$

$$[\hat{q}_i, \hat{q}_j^n] = 0 \equiv i\hbar\frac{\partial \hat{q}_j^n}{\partial \hat{p}_i}, \quad [\hat{p}_i, \hat{p}_j^n] = 0 \equiv -\,i\hbar\frac{\partial \hat{p}_j^n}{\partial \hat{q}_i}\,.$$

Using these definitions then for any $\hat{F}$ expressed as a power series in $\hat{q}$'s and $\hat{p}$'s it follows that

$$[\hat{q}_i, \hat{F}] \equiv i\hbar\,\partial\hat{F}/\partial\hat{p}_i \qquad \text{and} \qquad [\hat{p}_i, \hat{F}] \equiv -\,i\hbar\,\partial\hat{F}/\partial\hat{q}_i\,.$$

Choosing the special cases $\hat{F} = \hat{q}_i$ and $\hat{F} = \hat{p}_i$ leads to

$$i\hbar\dot{\hat{q}}_i = [\hat{q}_i, \hat{H}] = i\hbar\,\frac{\partial \hat{H}}{\partial \hat{p}_i}\,,\ i\hbar\dot{\hat{p}}_i = [\hat{p}_i, \hat{H}] = -\,i\hbar\,\frac{\partial \hat{H}}{\partial \hat{q}_i} \ \text{ and } \ \dot{\hat{H}} = \frac{\partial \hat{H}}{\partial t},$$

which we recognize is the operator form of Hamilton's equations

$$\dot{\hat{q}}_i = \frac{\partial \hat{H}}{\partial \hat{p}_i}\,,\ \dot{\hat{p}}_i = -\frac{\partial \hat{H}}{\partial \hat{q}_i} \ \text{ and } \ \dot{\hat{H}} = \frac{\partial \hat{H}}{\partial t}\,.$$

So we see that *the Heisenberg-picture operators satisfy the classical equations of motion*!

Relation between Quantum Mechanics and Classical Mechanics

(See Chapter 2, Sec. 2.5)

Relation to quantum mechanics: For a macroscopic action the classical path will dominate and so we can expand any dynamical operator $\hat{F} \equiv F(\hat{\vec{q}}(t), \hat{\vec{p}}(t), t)$ around this path using a Taylor expansion around the classical path $\left(\vec{q}_c(t), \vec{p}_c(t)\right)$. We then have

$$\hat{F} = F(\vec{q}_c, \vec{p}_c, t) + (\hat{\vec{q}} - \vec{q}_c) \cdot \frac{\partial F(\vec{q}_c, \vec{p}_c, t)}{\partial \vec{q}_c} + (\hat{\vec{p}} - \vec{p}_c) \cdot \frac{\partial F(\vec{q}_c, \vec{p}_c, t)}{\partial \vec{p}_c} + \cdots$$

Then for two such operators we can form their commutator using the canonical commutation relations and retain the nonvanishing $\mathcal{O}(\hbar)$ terms to give

$$[\hat{F}, \hat{G}] = \left[(\hat{\vec{q}} - \vec{q}_c) \cdot \frac{\partial F}{\partial \vec{q}_c}, (\hat{\vec{p}} - \vec{p}_c) \cdot \frac{\partial G}{\partial \vec{p}_c}\right] + \left[(\hat{\vec{p}} - \vec{p}_c) \cdot \frac{\partial F}{\partial \vec{p}_c}, (\hat{\vec{q}} - \vec{q}_c) \cdot \frac{\partial G}{\partial \vec{q}_c}\right] + \mathcal{O}(\hbar^2)$$

$$= i\hbar \left(\frac{\partial F}{\partial \vec{q}_c} \cdot \frac{\partial G}{\partial \vec{p}_c} - \frac{\partial F}{\partial \vec{p}_c} \cdot \frac{\partial G}{\partial \vec{q}_c} \right) + \mathcal{O}(\hbar^2) = i\hbar \{F, G\} + \mathcal{O}(\hbar^2) .$$

Note that: $\left[\hat{q}^i, \hat{p}^j\right] = i\hbar\delta^{ij}$, $\left[\hat{q}^i, \hat{q}^j\right] = \left[\hat{p}^i, \hat{p}^j\right] = 0$ and $\left\{q^i, p^j\right\} = \delta^{ij}$, $\left\{q^i, q^j\right\} = \left\{p^i, p^j\right\} = 0$.

The Correspondence Principle: We have then arrived at the *Correspondence Principle*, which is that

$$\lim_{\hbar \to 0} \frac{1}{i\hbar} [\hat{F}, \hat{G}] = \{F, G\}.$$

This is the basis of Dirac's *Canonical Quantization* approach, where the classical Hamiltonian $H(\vec{q}(t), \vec{p}(t), t)$ is replaced by the quantum Hamiltonian $\hat{H} \equiv H(\hat{\vec{q}}(t), \hat{\vec{p}}(t), t)$. This procedure is well defined up to possible ordering ambiguities of the $\hat{\vec{q}}(t)$ and $\hat{\vec{p}}(t)$. For singular systems the Correspondence Principle survives by *replacing the Poisson bracket with the Dirac bracket* (see Sec 2.9).

Interaction Picture in Quantum Mechanics

(See Chapter 4, Sec. 4.1.11)

Interaction picture: Consider a time-dependent Schrödinger picture Hamiltonian of the form

$$\hat{H}_s(t) = \hat{H}_0 + \hat{H}_{\rm int}(t) \,,$$

where H_0 is a time-independent ``free" part of the Hamiltonian and $H_{\rm int}(t)$ is a possibly time-dependent ``interaction" part of the Hamiltonian. The free and full evolution operators are respectively

$$\hat{U}_0(t'', t') \equiv e^{-i\hat{H}_0(t''-t')/\hbar} \quad \text{and} \quad \hat{U}(t'', t') \equiv Te^{-i\int_{t'}^{t''} dt\, \hat{H}_s(t)/\hbar} \,.$$

The Schrödinger picture state evolves time as $|\psi, t\rangle_s = \hat{U}(t, t_0)|\psi\rangle_h$, where the two pictures coincide at some arbitrary time t_0.

The relationship between the Schrödinger (s), interaction (I) and Heisenberg (h) pictures and the definition of the *interaction picture evolution operator* are

$$s \overset{U}{\to} h \quad \text{with} \quad s \overset{U_0}{\to} I \overset{U_I}{\to} h \quad \Rightarrow \quad \hat{U}(t, t_0) \equiv \hat{U}_0(t, t_0)\hat{U}_I(t, t_0) \quad \text{or} \quad \hat{U}_I(t, t_0) \equiv \hat{U}_0^\dagger(t, t_0)\hat{U}(t, t_0)$$

This defines the interaction picture in terms of the other two pictures,

$$\hat{A}_h(t) = \hat{U}(t, t_0)^\dagger \hat{A}_s \hat{U}(t, t_0) = \hat{U}_I(t, t_0)^\dagger \hat{A}_I(t) U_I(t, t_0) \quad \text{and} \quad |\psi\rangle_h = \hat{U}(t, t_0)^\dagger |\psi, t\rangle_s = \hat{U}_I(t, t_0)^\dagger |\psi, t\rangle_I \,.$$

So the interaction picture operators and state of the system are respectively given by

$$\hat{A}_I(t) = \hat{U}_0(t, t_0)^\dagger \hat{A}_s \hat{U}_0(t, t_0) = \hat{U}_I(t, t_0)\hat{A}_h(t)\hat{U}_I(t, t_0)^\dagger \quad \text{and}$$

$$|\psi, t\rangle_I = \hat{U}_0(t, t_0)^\dagger |\psi, t\rangle_s = \hat{U}_I(t, t_0)|\psi\rangle_h \,.$$

we have picture-independence of expectation values as we should, since

$$_s\langle\psi, t|\hat{A}_s|\psi, t\rangle_s = {}_h\langle\psi|\hat{A}_h(t)|\psi\rangle_h = {}_I\langle\psi, t|\hat{A}_I(t)|\psi, t\rangle_I \,.$$

Interaction Picture in Quantum Mechanics

(See Chapter 4, Sec. 4.1.11)

Define $\hat{H}_{\rm int}(t)$ in the interaction picture as $\hat{H}_I(t)$, then

$$\hat{H}_I(t) \equiv (\hat{H}_{\rm int})_I = \hat{U}_0(t,t_0)^\dagger \hat{H}_{\rm int}(t) \hat{U}_0(t,t_0) \; .$$

Recall the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H_s(t) |\psi(t)\rangle \; .$$

This can be written as

$$i\hbar \frac{d}{dt} (e^{-i\hat{H}_0(t-t_0)/\hbar} |\psi, t\rangle_I) = \{\hat{H}_0 + \hat{H}_{\rm int}(t)\} (e^{-i\hat{H}_0(t-t_0)/\hbar} |\psi, t\rangle_I) \; ,$$

and leads to

$$i\hbar \frac{d}{dt} |\psi, t\rangle_I = e^{+i\hat{H}_0(t-t_0)/\hbar} \hat{H}_{\rm int}(t) e^{-i\hat{H}_0(t-t_0)/\hbar} |\psi, t\rangle_I = \hat{U}_0(t,t_0)^\dagger \hat{H}_{\rm int}(t) \hat{U}_0(t-t_0) |\psi, t\rangle_I = \hat{H}_I(t) |\psi, t\rangle_I .$$

Using $|\psi, t\rangle_I = \hat{U}_I(t,t_0)|\psi\rangle_h$ we have

$$\frac{d}{dt} \hat{U}_I(t,t') = -(i/\hbar) \hat{H}_I(t) \hat{U}_I(t,t') \; ,$$

which leads to the interaction-picture version of the *Dyson formula*

$$\hat{U}_I(t'',t') = \hat{I} + (\tfrac{-i}{\hbar}) \int_{t'}^{t''} dt_1 \hat{H}_I(t_1) + (\tfrac{-i}{\hbar})^2 \int_{t'}^{t''} dt_1 \int_{t'}^{t_1} dt_2 \, \hat{H}_I(t_1)\hat{H}_I(t_2) + (\tfrac{-i}{\hbar})^3 \int_{t'}^{t''} dt_1 \int_{t'}^{t_1} dt_2 \int_{t'}^{t_2} dt_3 \, \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3) + \cdots$$

In a more compact form the interaction-picture Dyson formula is

$$\hat{U}_I(t'',t') = U_0(t'',t_0)^\dagger \hat{U}(t'',t') \hat{U}_0(t',t_0) = T\left(\exp\left\{ (-i/\hbar) \int_{t'}^{t''} dt \, \hat{H}_I(t) \right\} \right) \; ,$$

where T is the time-ordering operator that puts operators at later times to the left of those at earlier times.

Path integral approach to Quantum Mechanics

(See Chapter 4, Sec. 4.1.12)

Green's functions: Consider quantum mechanics with coordinate and momentum operators in the Heisenberg picture denoted as $\hat{q}_i(t)$ and $\hat{p}_i(t)$ respectively for $i = 1,\ldots,N$. The canonical commutation relations are

$$[\hat{q}_i(t), \hat{p}_j(t)] = [\hat{q}_i(t), \hat{p}_j(t)] = 0 \quad \text{and} \quad [\hat{q}_i(t), \hat{p}_j(t)] = i\hbar\delta_{ij} \, .$$

For simplicity consider a one-dimensional system. In the Schrödinger picture $\hat{q}\,|\,q\rangle_s = q\,|\,q\rangle_s$ and in the Heisenberg picture $\hat{q}(t)\,|\,q,t\rangle_h = q(t)\,|\,q,t\rangle_h$The amplitude for the system to evolve from q' at time t' to q'' at time t'' is called the *Green's function,* $G(q'',q';t'',t')$, of the system,

$$G(q'',q';t'',t') \equiv {}_h\langle q'',t''\,|\,q',t'\rangle_h = {}_s\langle q''\,|\,\hat{U}(t'',t')\,|\,q'\rangle_s \, .$$

Now dividing $(t''-t')$ into $(n+1)$ equal intervals of length

$$\delta t = (t''-t')/(n+1) \quad \text{with} \quad t_j \equiv t' + j\delta t \qquad \text{for} \qquad j = 0,1,\cdots,n+1$$

where we see that $t_{j+1} = t_j + \delta t$, $t'' = t_{n+1}$ and $t' = t_0$. We omit the labels h and s for brevity from now on in this section. Noting the completeness of the coordinate-space and momentum-space bases in both the Heisenberg and Schrödinger pictures,

$$\int dq\,|\,q,t\rangle\langle q,t\,| = \int dq\,|\,q\rangle\langle q\,| = \int dp\,|\,p,t\rangle\langle p,t\,| = \int dp\,|\,p\rangle\langle p\,| = \hat{I}$$

we can write

$$\langle q'',t''\,|\,q',t'\rangle = \int dq_1 \cdots \int dq_n\,\langle q'',t''\,|\,q_n,t_n\rangle \cdots \langle q_2,t_2\,|\,q_1,t_1\rangle\langle q_1,t_1\,|\,q',t'\rangle \, .$$

Since $\hat{U}(t'',t') = e^{-i\hat{H}(t''-t')/\hbar}$ and we can write

$$\langle q_{j+1},t_{j+1}\,|\,q_j,t_j\rangle = \langle q_{j+1}\,|\,e^{-i\delta t\hat{H}/\hbar}\,|\,q_j\rangle = \delta(q_{j+1}-q_j) - i(\delta t/\hbar)\langle q_{j+1}\,|\,\hat{H}\,|\,q_j\rangle + O(\delta t^2) \, .$$

Using $\langle q\,|\,p\rangle = (1/\sqrt{2\pi\hbar})e^{ipq/\hbar}$ leads to

$$\begin{aligned}\langle q_{j+1}\,|\,\hat{H}\,|\,q_j\rangle &= \langle q_{j+1}\,|\,H(\hat{q},\hat{p})\,|\,q_j\rangle = \int dp_j\,\langle q_{j+1}\,|\,p_j\rangle\langle p_j\,|\,H(\hat{q},\hat{p})\,|\,q_j\rangle \\ &= \int dp_j\,H'(q_j,p_j)\langle q_{j+1}\,|\,p_j\rangle\langle p_j\,|\,q_j\rangle = \int (dp_j/2\pi\hbar)e^{ip_j(q_{j+1}-q_j)/\hbar}\,H(q_j,p_j) \, .\end{aligned}$$

Path integral approach to Quantum Mechanics

(See Chapter 4, Sec. 4.1.12)

Hence we can write

$$\langle q_{j+1}, t_{j+1} | q_j, t_j \rangle = \int (dp_j/2\pi\hbar) e^{ip_j(q_{j+1}-q_j)/\hbar} \left\{ 1 - i(\delta t/\hbar) H(q_j, p_j) + \mathcal{O}(\delta t^2) \right\}$$
$$= \int (dp_j/2\pi\hbar) e^{i(\delta t/\hbar)[p_j\{(q_{j+1}-q_j)/\delta t\} - H(q_j,p_j)]} + \mathcal{O}(\delta t^2) \,.$$

Using this result gives

$$\langle q'', t'' | q', t' \rangle = \lim_{n\to\infty} \int \prod_{i=1}^n dq_i \int \prod_{k=0}^n (dp_k/2\pi\hbar) e^{i\sum_{j=0}^n \delta t[p_j\{(q_{j+1}-q_j)/\delta t\} - H(q_j,p_j)]/\hbar}$$
$$\equiv \int \mathcal{D}q\, \mathcal{D}p\, e^{(i/\hbar)\int_{t'}^{t''} dt[p\dot{q} - H(q,p)]}$$

The last line of is shorthand notation for the line above and represents an integration over all possible *paths in phase space* with the coordinate endpoints $q(t'') = q''$ and $q(t') = q'$. There are no restrictions on the momentum endpoints.

Consider a time-independent Hamiltonian of the form $\hat{H} \equiv H(\hat{p}, \hat{q}) = (\hat{p}^2/2m) + V(\hat{q})$ then using

$$\int \frac{dp}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar}\delta t \left[p\dot{q} - \frac{p^2}{2m} \right] \right\} \to \int \frac{dp}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar}\delta t \left[p\dot{q} - (1 - i\epsilon)\frac{p^2}{2m} \right] \right\}$$
$$= \sqrt{\frac{m}{2i\pi\hbar\delta t(1 - i\epsilon)}} \exp\left\{ \frac{i}{\hbar}\delta t(1 + i\epsilon) \left[\frac{1}{2} m\dot{q}^2 \right] \right\},$$

where $\epsilon \to 0^+$ provides an infinitesimal damping term to define the integral.

Path integral approach to Quantum Mechanics

(See Chapter 4, Sec. 4.1.12)

Hence we have

$$\langle q'', t'' | q', t' \rangle = \lim_{n\to\infty} \left(\frac{m}{(2i\pi\hbar\delta t)(1 - i\epsilon)} \right)^{(n+1)/2} \int \prod_{i=1}^{n} dq_i$$
$$\times \exp\left\{ \frac{i}{\hbar}\delta t \sum_{j=0}^{n} \left[(1 + i\epsilon)\frac{m}{2}\left(\frac{q_{j+1} - q_j}{\delta t}\right)^2 - (1 - i\epsilon)V(q_j) \right] \right\}.$$

This is conventionally expressed in the compact form

$$\langle q'', t'' | q', t' \rangle = \int \mathscr{D}q \, \exp\left\{ \frac{i}{\hbar} S[q] \right\} = \int \mathscr{D}q \, \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} dt \, L(q, \dot{q}) \right\}$$

where $L(q, \dot{q})$ is the usual Lagrangian for the form of Hamiltonian that we are considering plus a damping term,

$$L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - V(q) + i\epsilon(\frac{1}{2}m\dot{q}^2 + V(q)),$$

where $\epsilon \to 0^+$ provides an infinitesimal damping term to help define the integral. This is essential and leads to the *Feynman boundary conditions* in QFT as we will later see. We can always remove an infinite constant so that $V(q) > 0$. We do not fuss here about the formal mathematical existence of the continuum limit, $\lim(n \to \infty)$, of single path integrals as we will later see that only ratios of path integrals are physically important. It is the continuum limit of such ratios that is physically relevant.

We can also understand the damping term as the result of formulating the theory in Euclidean space $(t'' - t') \to -i(\tau'' - \tau') \equiv -i\hbar\beta$. This replaces the evolution operator for a time-independent Hamiltonian according to

$$\exp(-i\hat{H}(t'' - t')/\hbar) \to \exp(-(\tau'' - \tau')\hat{H}/\hbar) = \exp(-\beta\hat{H}) \, .$$

Repeating all steps in Euclidean space leads to non=oscillatory damped integrals. Then rotating back to Minkowski space leads to a residual term that is precisely the above damping term.

Spectral function and the partition function of statistical mechanics

(See Chapter 4, Sec. 4.1.12)

Spectral function: The spectral function, denoted F here, is defined as the trace of the evolution operator

$$F(t''-t') \equiv \mathrm{tr}\left(e^{-i\hat{H}(t''-t')/\hbar}\right) = \sum_n \langle n|e^{-i\hat{H}(t''-t')/\hbar}|n\rangle = \sum_n e^{-iE_n(t''-t')/\hbar}$$

$$= \int dq\, {}_s\langle q|e^{-i\hat{H}(t''-t')/\hbar}|q\rangle_s = \int dq\, {}_h\langle q, t''|q, t'\rangle_h = \int dq\, G(q, q; t'', t')$$

$$= \int \mathscr{D}p\, \mathscr{D}q_{(\mathrm{periodic})} \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''} dt\, \left[p\dot{q} - H(p, q)\right]\right\} = \int \mathscr{D}q_{(\mathrm{periodic})} \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''} dt\, L(q, \dot{q})\right\}$$

where the integration is over all periodic paths in coordinate space with period $(t''-t')$. There is no similar periodicity requirement in momentum space. Here we have again assumed $\hat{H} = (\hat{p}^2/2m) + V(\hat{q})$ so that $L = \frac{1}{2}m\dot{q}^2 - V(q)$.

Partition function: The partition function of statistical mechanics is

$$Z(\beta) = \sum_{n=0}^{\infty} \exp(-\beta E_n) = \mathrm{tr} e^{-\beta\hat{H}} = {}_s\langle q|e^{-\beta\hat{H}}|q\rangle_s = \int \mathscr{D}p\, \mathscr{D}q_{(\mathrm{periodic})}\, e^{\frac{1}{\hbar}\int_{\tau'}^{\tau''} d\tau\, [ip\dot{q} - H(p,q)]}$$

$$= \int \mathscr{D}p\, \mathscr{D}q_{(\mathrm{periodic})} \exp\left\{\frac{1}{\hbar}\int_{\tau'}^{\tau''} d\tau\, \left[ip\dot{q} - H(p, q)\right]\right\} = \int \mathscr{D}q_{(\mathrm{periodic})} \exp\left\{-\frac{1}{\hbar}\int_{\tau'}^{\tau''} d\tau\, L_{\mathrm{E}}(q, \dot{q})\right\},$$

where the Euclidean Lagrangian is $L_E = \frac{1}{2}m\dot{q}^2 + V(q)$ with $\dot{q} \equiv dq/d\tau$.

Time-ordered products, sources and generating functionals

(See Chapter 4, Sec. 4.1.12)

Consider now adding an external source term to the Hamiltonian operator such that

$$\hat{H}^J(t) \equiv H - J(t)\hat{q} = (\hat{p}^2/2m) + V(\hat{q}) - J(t)\hat{q}$$

which leads to the source-dependent evolution operator in the Schrödinger picture,

$$\hat{U}^J(t'',t') = T e^{-(i/\hbar)\int_{t'}^{t''} dt\,[\hat{H} - J(t)\hat{q}]}.$$

Then we can go to an interaction picture by identifying

$$\hat{H}(t) \to \hat{H}^J(t)\,, \quad \hat{H}_0 \to \hat{H}\,, \quad \hat{U}(t'',t') \to \hat{U}^J(t'',t')\,, \quad \hat{U}_0(t'',t') \to \hat{U}(t'',t') = e^{-i\hat{H}(t''-t')}\,,$$

$$\hat{H}_{\text{int}}(t) = -J(t)\hat{q}\,, \quad \hat{H}_I(t) = \hat{U}(t,t_0)^\dagger \hat{H}_{\text{int}}(t)\hat{U}(t,t_0) = -J(t)\hat{q}_I(t)\,,$$

and $\hat{U}_I(t'',t') = T e^{-(i/\hbar)\int_{t'}^{t''} dt\, H_I(t)} = T e^{(i/\hbar)\int_{t'}^{t''} dt\, J(t)\hat{q}_I(t)}$.

The source-dependent Green's function is then

$$ {}_h\langle q'',t''|q',t'\rangle_h^J = {}_s\langle q''|T e^{-(i/\hbar)\int_{t'}^{t''} dt[\hat{H} - J(t)\hat{q}]}|q'\rangle_s = {}_I\langle q'',t''|T e^{(i/\hbar)\int_{t'}^{t''} dt\, J(t)\hat{q}_I(t)}|q',t'\rangle_I$$

$$= \int \mathcal{D}p\,\mathcal{D}q\, e^{(i/\hbar)\int_{t'}^{t''} dt\left[p\dot{q} - H(p,q) + J(t)q(t)\right]} = \int \mathcal{D}q\, e^{(i/\hbar)\int_{t'}^{t''} dt\left[L(q,\dot{q}) + J(t)q(t)\right]}.$$

The spectral function becomes

$$F^J(t'',t') = \text{tr}\{\hat{U}^J(t'',t')\} = \int \mathcal{D}q_{(\text{periodic})} e^{(i/\hbar)\int_{t'}^{t''} dt\left[L(q,\dot{q}) + J(t)q(t)\right]}.$$

Using the definition of the functional derivative it follows that

$$(-i\hbar)^k \frac{\delta^k}{\delta J(t_1)\cdots\delta J(t_k)}\,\hat{U}_I(t'',t')\Bigg|_{J=0} = T\left(\hat{q}_I(t_1)\cdots\hat{q}_I(t_k)\right)\Big|_{J=0} = T\left(\hat{q}_h(t_1)\cdots\hat{q}_h(t_k)\right).$$

Time-ordered products, sources and generating functionals

(See Chapter 4, Sec. 4.1.12)

We take the source to zero after the derivatives are taken and we have recognized that $\hat{H}_{\rm int} \to 0$ when $J \to 0$ and so $\hat{q}_I(t) \to \hat{q}_h(t)$. We observe that

$$ {}_h\langle q'', t'' | T\left[\hat{q}_h(t_1)\cdots\hat{q}_h(t_k)\right] | q', t'\rangle_h = (-i\hbar)^k \frac{\delta^k}{\delta J(t_1)\cdots\delta J(t_k)} \, {}_h\langle q'', t'' | q', t'\rangle_h^J \Big|_{J=0} $$

$$ = (-i\hbar)^k \frac{\delta^k}{\delta J(t_1)\cdots\delta J(t_k)} \int \mathscr{D}q \, e^{(i/\hbar)\int_{t'}^{t''} dt\,\left[L(q,\dot{q}) + J(t)q\right]} \Big|_{J=0} $$

$$ = \int \mathscr{D}q \, q(t_1)\cdots q(t_k) \exp\left\{ (i/\hbar) \int_{t'}^{t''} dt \, L(q,\dot{q}) \right\} , $$

where recall the understood boundary conditions $q(t'') = q''$ and $q(t') = q'$.

Now consider the consequence of taking $T = t'' = -t'$ and the limit $T \to \infty(1 - i\epsilon)$, which corresponds to formulating in Euclidean space and then rotating back to Minkowski space. We see that

$$ F^J(T, -T) = \mathrm{tr}\{\hat{U}^J(T, -T)\} = \sum_n {}_s\langle E_n | \hat{U}^J(T, -T) | E_n\rangle_s $$

$$ = \sum_n {}_s\langle E_n | \hat{U}(T, t_0)\hat{U}_I(T, -T)\hat{U}(-T, t_0)^\dagger | E_n\rangle_s = \sum_n e^{-iE_n(2T)/\hbar} \, {}_s\langle E_n | \hat{U}_I(T, -T) | E_n\rangle_s $$

and so taking the limit $T \to \infty(1 - i\epsilon)$ suppresses all contributions in the sum relative to that from the ground state $|\Omega\rangle = |E_0\rangle_s$. Define $F(T, -T) \equiv F^{J=0}(T, -T) = \sum_n e^{-iE_n(2T)/\hbar}$. Then finally we can define the generating functional $Z[J]$ as

$$ Z[j] \equiv \lim_{T\to\infty(1-i\epsilon)} \frac{F^j(T, -T)}{F(T, -T)} = \lim_{T\to\infty(1-i\epsilon)} \langle \Omega | \hat{U}_I(T, -T) | \Omega\rangle \, . $$

Time-ordered products, sources and generating functionals

(See Chapter 4, Sec. 4.1.12)

Equivalent ways of writing the generating functional: Combining all of the results we see that the generating functional $Z[J]$ can be written in a variety of ways,

$$Z[j] \equiv \lim_{T\to\infty(1-i\epsilon)} \frac{F^j(T,-T)}{F(T,-T)} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\{\hat{U}^j(T,-T)\}}{\mathrm{tr}\{\hat{U}(T,-T)\}} = \lim_{T\to\infty(1-i\epsilon)} \frac{\mathrm{tr}\left[Te^{-i\int_{-T}^{T}dt\,[\hat{H}-J\hat{q}]}\right]}{\mathrm{tr}\left[Te^{-i\int_{-T}^{T}dt\,\hat{H}}\right]}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \langle\Omega|\hat{U}_I(T,-T)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}q_{(\text{periodic})}\, e^{(i/\hbar)\{S[q]+\int_{-T}^{T}dt\,Jq\}}}{\int \mathcal{D}q_{(\text{periodic})}\, e^{(i/\hbar)S[q]}}\,.$$

Vacuum expectation values of time-ordered products: Making use of the earlier result of differentiating $\hat{U}_I(t'',t')$ with respect to the source we finally arrive at a key result that we will exploit when we generalize to QFT, which is that

$$\langle\Omega|T\hat{q}_h(t_1)\ldots\hat{q}_h(t_k)|\Omega\rangle = (-i\hbar)^k \frac{\delta^k}{\delta J(t_1)\cdots\delta J(t_k)} Z[J]\bigg|_{J=0}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}q_{(\text{periodic})}\, q(t_1)\cdots q(t_k)\, e^{(i/\hbar)S[q]/\hbar}}{\int \mathcal{D}q_{(\text{periodic})}\, e^{(i/\hbar)S[q]}}\,.$$

Important lesson: The vacuum, expectation value of the time-ordered products of Heisenberg-picture operators can be obtained from the generating functional and can be understood in terms of the path (or functional) integral in terms of the action, where we integrate over paths (functions) periodic in time.