The Standard Model of Particle Physics

Lecture 2

Optical Theorem

The $\mathbb{S}$ matrix can be written as

$$\mathbb{S} = \mathbb{1} + i\mathbb{T} \tag{1.1}$$

This matrix is unitary, so

$$\mathbb{S}\mathbb{S}^\dagger = (\mathbb{1} + i\mathbb{T})(\mathbb{1} - i\mathbb{T}^\dagger) = \mathbb{1} \tag{1.2}$$

$$0 = i(\mathbb{T}_{ij} - \mathbb{T}^\dagger_{ij}) + \mathbb{T}_{il}\mathbb{T}_{lj} \tag{1.3}$$

The $\mathbb{T}$ matrix is infinite dimensional, with each row for each set of particles and for each momentum k^μ. We can relate this to the $\mathcal{M}$ matrix by

$$\mathbb{T}_{ij} = \mathcal{M}_{ij}(2\pi)^4\delta^4(\sum p_i - \sum p_j) \tag{1.4}$$

and replacing the sum over the index l by

$$\sum_l \to \sum_l \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \tag{1.5}$$

We get

$$-i(\mathcal{M}_{ij} - M^\dagger_{ji})\delta^4(p_i - p_j) = \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{lj}\delta^4(p_i - p_l)(2\pi)^4\delta^4(p_l - p_j) \tag{1.6}$$

$$= \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{lj}\delta^4(p_i - p_j)(2\pi)^4\delta^4(p_l - p_j) \tag{1.7}$$

Optical Theorem

and so

$$(-i(\mathcal{M}_{ij} - M^\dagger_{ji}) - \prod_{p_l} \frac{d^3 p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{lj}(2\pi)^4\delta^4(p_l - p_j))\delta^4(p_i - p_j) = 0 \tag{1.8}$$

We now choose the case $i = j$. The overall delta function is not zero, so what is in the brackets must be zero. We get

$$-i(\mathcal{M}_{ii} - M^\dagger_{ii}) - = \prod_{p_l} \frac{d^3 p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{li}(2\pi)^4\delta^4(p_l - p_j) \tag{1.9}$$

$$2Im[\mathcal{M}_{ii}] = |\mathcal{M}_{il}|^2 d\Phi_{LIPS} \tag{1.10}$$

where the sum over all possible l states is assumed.

Now we remind the equation to calculate a cross section

$$d\sigma = \frac{1}{2E_1 2E_2|v_1 - v_2|}|\bar{\mathcal{M}}|^2 d\Phi_{LIPS} \tag{1.11}$$

The equation for a (partial) decay width (inverse of decay time) is

$$d\Gamma = \frac{1}{2M}|\bar{\mathcal{M}}|^2 d\Phi_{LIPS} \tag{1.12}$$

We get

$$Im[\mathcal{M}_{ii}] = M\Gamma \tag{1.13}$$

Where we have "assumed" $|\bar{\mathcal{M}}|^2 = |\mathcal{M}|^2$. This is true for scalar particles. We will see that this relation holds also for other particles. Such particles will have spinor or Lorentz indices in their self energy, and correctly accounting for them will keep such relation true.

Optical Theorem

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Lorentz Invariant Phase Space

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Ward Identity in QED and QCD

When a process contains one or more external photons, the amplitude is proportional to the external photon polarization, that can be factorized:

$$\mathcal{M} = \mathcal{M}_\mu \varepsilon^\mu(k)$$

The Ward identity tells us that by replacing the polarization vector with the photon momentum, we get zero

$$\mathcal{M}_\mu k^\mu = 0$$

This tells us that the amplitude for a longitudinally polarised photon is zero for example.

In QCD, if we check the Ward identity for the process $q\bar{q} \to gg$ we notice that it is violated.

Indeed one can check that there are unphysical polarizations for which the amplitude does not cancel.

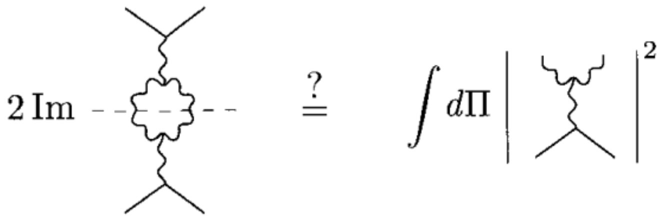
Optical Theorem violation, BRST symmetry and ghosts

This leads to a violation of the optical theorem when considering loop diagram $q\bar{q} \to q\bar{q}$

This happens due to an issue that arises when quantising non-abelian gauge theories.

This is solved by introducing additional fields (ghosts) and the BRST symmetry.

(See QCD course for more details)



Running Coupling Constants and Beta Functions in QED

$$\beta_e(e) = M\frac{\partial e}{\partial M} = eM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right) = \frac{e^3}{12\pi^2}$$

$$\beta_\alpha = M\frac{\partial \alpha}{\partial M} = \frac{e}{2\pi}\beta_e = \frac{e^4}{24\pi^3} = \frac{2}{3\pi}\alpha^2$$

$$\frac{d\alpha}{d\log M^2} = \frac{1}{2}\beta_\alpha = \frac{1}{3\pi}\alpha^2$$

We can now solve this differential equation

$$\frac{d\alpha}{\alpha^2} = \frac{1}{3\pi}d\log M^2$$

$$[\frac{1}{\alpha}]^{\alpha(q^2)}_{\alpha(M_0^2)} = \frac{1}{3\pi}\log\frac{q^2}{M_0^2}$$

$$\alpha(q^2) = \frac{\alpha(M_0^2)}{1 - \frac{\alpha(M_0^2)}{3\pi}\log\frac{q^2}{M_0^2}}$$

The Beta function is positive, so the coupling constant grows going towards larger energy scales, and hits a Landau Pole at very large energy

Running Coupling Constants and Beta Functions in QCD

$$\beta(g) = gM\frac{\partial}{\partial M}\left(-\delta_1+\delta_2+\frac{1}{2}\delta_3\right) \quad (3.220)$$

$$= \frac{g^3}{(4\pi)^2}\Gamma(2-\frac{d}{2})(-2)(2-\frac{d}{2})\left(C_2(r)+C_2(G)-C_2(r)+\frac{1}{2}(\frac{5}{3}C_2(G)-\frac{4}{3}n_fC(r))\right) \quad (3.221)$$

$$= -\frac{g^3}{(4\pi)^2}\Gamma(3-\frac{d}{2})\left(\frac{11}{3}C_2(G)-\frac{4}{3}n_fC(r)\right) \quad (3.222)$$

$$= -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G)-\frac{4}{3}n_fC(r)\right) \quad (3.223)$$

Fr an abelian group, $C_2(G)=0$ and $\beta>0$. FOr a non abelian group, $\beta(g)<0$ for small n_f. For the QCD case

$$C_2(G) = N = 3 \quad (3.224)$$

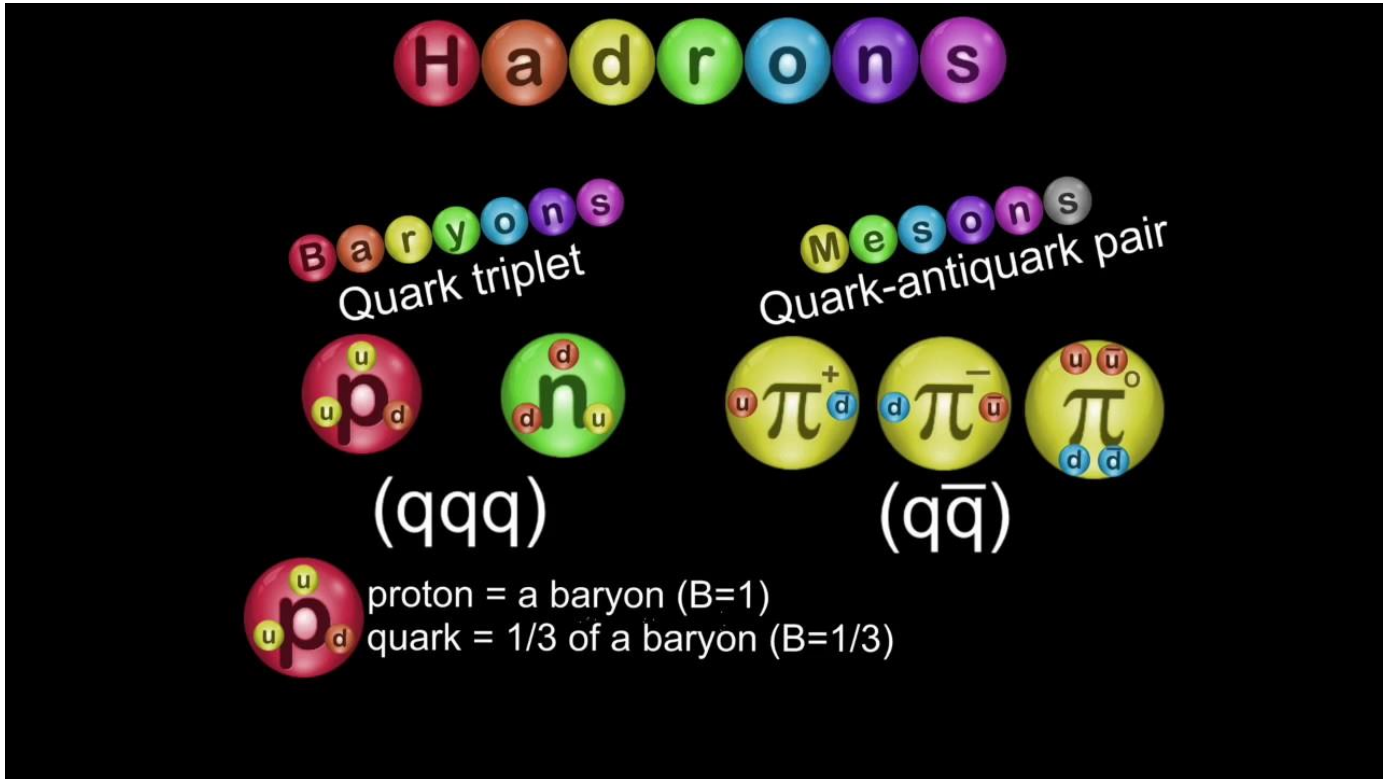
$$C(F) = \frac{1}{2} \quad (3.225)$$

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(11-\frac{2}{3}n_f\right) \quad (3.226)$$

The Beta function is negative, so the coupling constant grows going towards smaller energy scales, and hits a Landau Pole at $\Lambda_{QCD} \approx 330MeV$

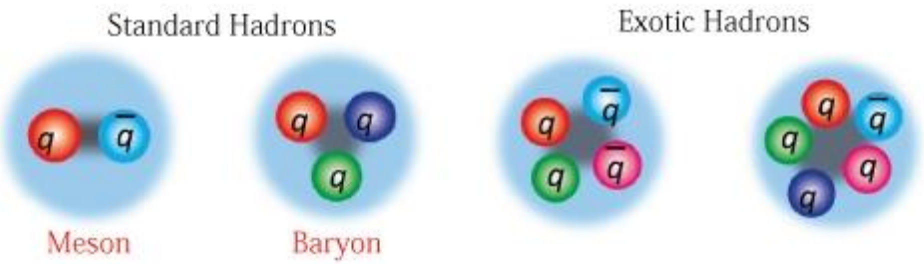
Confinement, Baryons, Mesons

As the strong coupling constant becomes very large at low energies, perturbative expansion fails. However, one can get the order of magnitude of the important energy scale $\Lambda_{QCD} \approx 330 MeV$

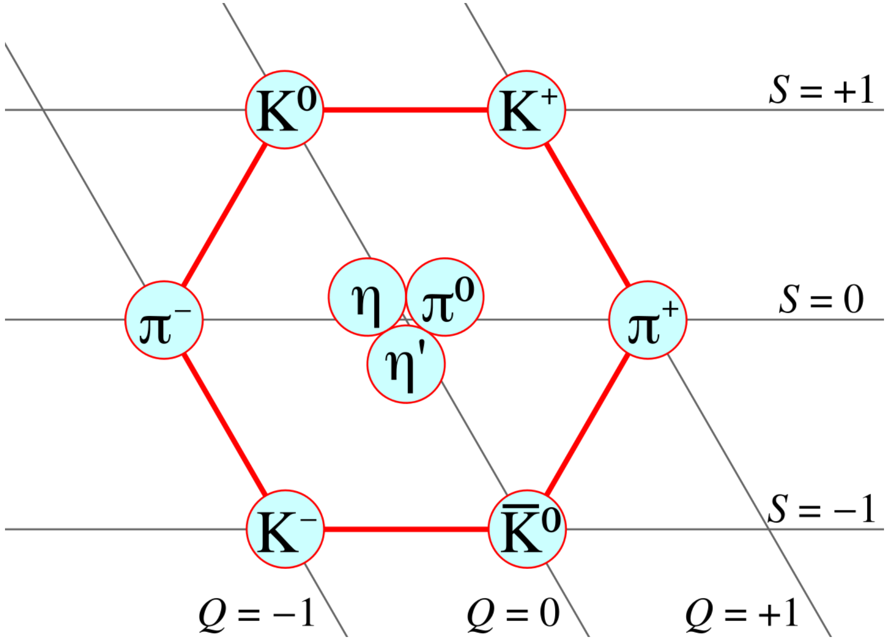


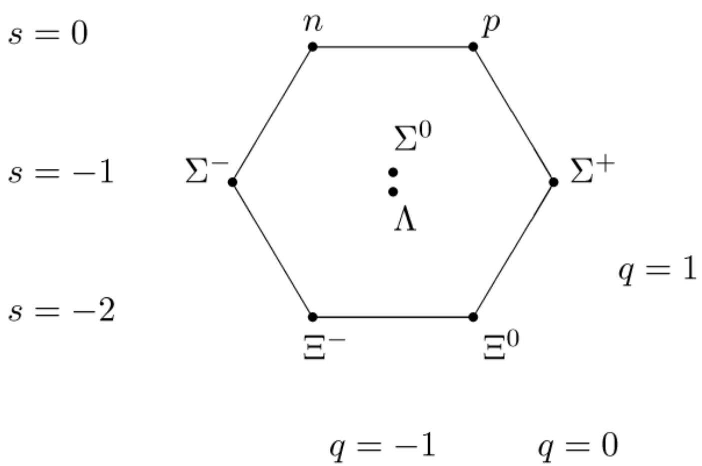
Color Singlets

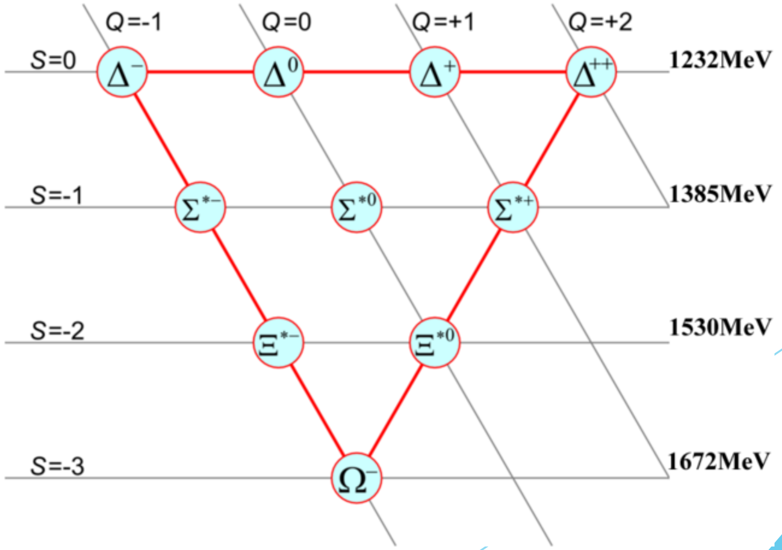
- Quarks confine to form color singlets
- Possible ways to form singlets:
- $3 \otimes \bar{3} = 1 \oplus 8$
- $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$
- $3 \otimes \bar{3} \otimes 3 \otimes \bar{3} = 1 \oplus 8 \oplus 8 \oplus (8 \otimes 8) = 1 \oplus 8 \oplus 8 \oplus 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27$
- $3 \otimes 3 \otimes 3 \otimes 3 \otimes \bar{3} = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 8 \oplus (8 \otimes 8) \oplus (8 \otimes 8) \oplus (10 \otimes 8)$



Symmetry Patterns for Baryons and Mesons







Goldstone Theorem

Let Φ be a real scalar field in the fundamental repr. of $SO(N)$. a Φ^4 theory invariant under $SO(N)$ can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2}\mu^2 \Phi^\dagger \Phi - \frac{\lambda}{4}(\Phi^\dagger \Phi)^2 \tag{4.15}$$

You may notice that the mass term has the wrong sign. No worries! We will see that the theory still makes perfectly sense. What will happen is a symmetry breaking, and some fields will need to be rewritten, as some of them will get a nonzero vev. Taking the potential, we find the minima condition

$$\frac{\partial V}{\partial \Phi} = -\mu^2 \Phi^\dagger + \lambda(\Phi^\dagger \Phi)\Phi^\dagger = 0 \tag{4.16}$$

$$\rightarrow \Phi^\dagger \Phi = \frac{\mu^2}{\lambda} \tag{4.17}$$

Thus we need to have at least one component of the n-vector that has nonzero vev. Note that we can take to be just one component without loss of generality, because we can always rotate the vector by a $SO(N)$ transformation to have all but one components with no vev. This transformation will define the physical base. Thus

$$\langle \Phi \rangle = (0, ..., \sqrt{\frac{\mu^2}{\lambda}}) \tag{4.18}$$

Goldstone Theorem

From now on, we are not more allowed to perform a generic $SO(N)$ rotation, as the system is clearly not invariant anymore under such transformations. What remains is an $SO(N-1)$ symmetry for the first $N-1$ components of Φ. We will call them $\tilde{\Phi}$, while we will call ρ the last field component of Φ,

$$\Phi = (\tilde{\Phi}, \sqrt{\frac{\mu^2}{\lambda}} + \rho) \tag{4.19}$$

If we rewrite the lagrangian using such fields we get

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho \\ & + \frac{1}{2}\mu^2\left(\tilde{\Phi}^\dagger\tilde{\Phi} + (\frac{\mu^2}{\lambda}\rho)^2\right) \\ & + \frac{1}{4}\lambda\left(\tilde{\Phi}^\dagger\tilde{\Phi} + (\frac{\mu^2}{\lambda}\rho)^2\right)^2 \end{aligned} \tag{4.20}$$

$$= \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}(2\mu^2)\rho^2 + \mathcal{O}(S^3) \tag{4.21}$$

The field ρ now has a mass term with the right sign, while the remaining fields are massless and have e remnant $SU(N-1)$ symmetry!

Goldstone Theorem

Goldstone Theorem: If a Lagrangian, invariant under a set of continuus transformations, gets the symmetry spontaneously broken, there is a massless goldstone boson for each broken generator.

How many generators where broken?

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N - 1 \tag{4.22}$$

that is exactly the number of massless particles remaining in the theory!

PROOF We want to proove that, $\forall$ continuus symmetries of $\mathcal{L}$ that are not a symmetry of the minimum, then

$$M_{ab} = \frac{\partial V}{\partial \phi_a \partial \phi_b}(\phi_0) \tag{4.23}$$

has a zero eigenvalue. The symmetry can be expressed as a infinitesimal tranformation

$$\phi^a \to \phi^a + \alpha R^{ab} \phi^b, \quad \alpha \ll 1 \tag{4.24}$$

We take ϕ as constant, so a symmetry of $\mathcal{L}$ is a symmetry of V. The invariance of V means that

$$V(\phi^a) = V(\phi^a + \Delta\phi^a) \tag{4.25}$$

$$\frac{\partial V}{\partial \phi^a}(\phi)\Delta\phi^a(\phi) = \frac{\partial V}{\partial \phi^a}(\phi)\alpha R^{ac}\phi^c = 0 \tag{4.26}$$

Goldstone Theorem

We differentiate the previous relation w.r.t ϕ_b:

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi)R^{ac}\phi^c + \frac{\partial V}{\partial\phi^a}(\phi)R^{ab} = 0 \tag{4.27}$$

we now specify to the point $\phi = \phi_0$,

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi_0)R^{ac}\phi_0^c + \frac{\partial V}{\partial\phi^a}(\phi_0)R^{ab} = 0 \tag{4.28}$$

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi_0)R^{ac}\phi_0^c = M_{ab}R^{ac}\Phi_0^c = 0 \tag{4.29}$$

where the second equation is obtained using the fact that the gradient of the potential is zero at the minima. Now, the number of nonzero components $R^{ac}\phi_0^c$ is equal to the number of broken generators R^{ac}, i.e. generators for which

$$R^{ac}\Phi^c \neq 0 \tag{4.30}$$

These are all linearly independent components, and as all components of

$$M_{ab}R^{ac}\Phi_0^c \tag{4.31}$$

are required to be zero, it follows that the matrix M_{ab} needs to have the same number of zero eigenvalues as the number of broken generators.

Chiral Symmetry of massless QCD

We want to study symmetries of YM theories, for example QCD

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + (\bar{u}, \bar{d}, \bar{s})i\not{D}\begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{4.5}$$

We can rewrite the lagrangian decomposing in left and right handed fields

$$q_i = \left(\left(\frac{1+\gamma^5}{2}\right) + \left(\frac{1-\gamma^5}{2}\right)\right) q_i = \left(\frac{1+\gamma^5}{2}\right) q_i + \left(\frac{1+\gamma^5}{2}\right) q_i = q_{i,R} + q_{i,L} \tag{4.6}$$

We get the following lagrangian

$$(\bar{u}, \bar{d}, \bar{s})i\not{D}\begin{pmatrix} u \\ d \\ s \end{pmatrix} = (\bar{u}_R, \bar{d}_R, \bar{s}_R)i\not{D}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} + (\bar{u}_L, \bar{d}_L, \bar{s}_L)i\not{D}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \tag{4.7}$$

The left and right fields do not talk to each other. Such a theory is invariant under the global symmetry of the group

$$U(3)_L \otimes U(3)_R = U(1)_L \otimes U(1)_R \otimes SU(3)_L \times SU(3)_R \tag{4.8}$$

Chiral Symmetry of massless QCD

The $U(1)$ factors transform all fields of the given helicity and does not act on the fields of the other helicity. The $SU(3)$ factors apply $SU(3)$ transformations to the fields of the given helicity (thus mixing up different flavours) and do not act on the fields of the other helicity.

$$q_L \to e^{i\alpha} q_L \quad U(1)_L \tag{4.9}$$

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \quad SU(3)_L \tag{4.10}$$

The symmetry group can be rewritten as

$$U(3)_V \otimes U(3)_A = U(1)_V \otimes U(1)_A \otimes SU(3)_V \times SU(3)_A \tag{4.11}$$

$U(1)_V$ acts on all quarks with same phase shift

$$q \to e^{i\alpha} q \tag{4.12}$$

This symmetry is exact and is called $U(1)_B$ barion number conservation. The $SU(3)_V$ is an $SU(3)$ transformation acting in the same way for left and right handed components

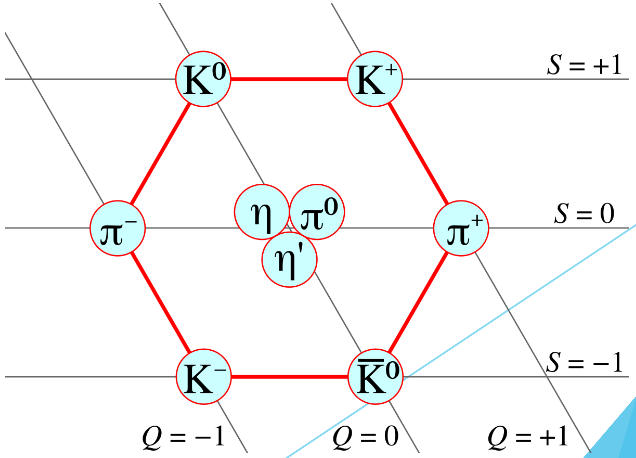
$$Q \to UQ \tag{4.13}$$

$$Q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{4.14}$$

This symmetry would be exact even with quark masses turned on, as long as they are all degenerate. So it is broken by quark mass differences, $m_d - m_u$, $2m_s - m_d - m_u$. This (approximate) symmetry is seen in nature and is called $SU(3)_F$. The subgroup with only the u, d quarks is called $SU(2)_F$ or isospin symmetry.

Chiral Symmetry of massless QCD

- The $SU(3)_A$ symmetry gets spontaneously broken by nonperturbative effects of QCD vacuum
- This means 8 broken generators, so one expects 8 massless goldstone bosons
- However quarks are not really massless, they have a small mass, (much) smaller than Λ_{QCD}
- This means the $SU(3)_A$ symmetry is just an approximate symmetry, and as such we expect 8 nearly-massless (=very light) goldstone bosons
- These are indeed the mesons of the meson octet!



Chiral Symmetry of massless QCD: The U(1) problem

- The $U(1)_A$ symmetry would gets spontaneously broken by nonperturbative effects of QCD vacuum as well
- This would mean that one would expect an additional light meson, that was found, but was not light!
- This originated the so called “$U(1)$” problem
- The solution to the problem is the fact that the $U(1)_A$ symmetry is a symmetry of the classical lagrangian, but gets broken by quantum effects, and therefore is not a symmetry at all.
- As such, the Goldstone theorem does not apply!