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The Standard Model Lagrangian (Lecture 4)

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Abstract.

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1 Yukawa sector in the SM

So far our model is still pretty simple, in the sense that depends on a very limited number of parameters: the 3 gauge couplings $g_{1,2,3}$, the higgs self interaction coupling λ, and the higgs mass parameter μ^2, for a total of 5 parameters. However, we still need to explain all fermion masses, that will require at least 9 more parameters. We will see that we need to add indeed 13 new parameters. This will open the discussion about flavour.

1.1 Yukawa Lepton Sector

We go back to the fermion part of the lagrangian. So far, we have only included the kinetic term with the covariant derivative. this does not include a mass term for the fermions, unfortunately. We start from leptons, the lagrangian reads[1]

$$\mathcal{L} = \bar{L}_L i \not{D} L_L + \bar{e}_R i \not{D} e_R \tag{1.1}$$

The fields L_L and e_R are not just an $SU(2)_L$ doublet and singlet respectively, they are also in a 3 representation of 2 a (approximate) global symmetry groups, $U(3)_L$, $U(3)_e$. As the operator $\not{D}$ is the identity in flavour space, it is trivial to see that this part of the lagrangian is invariant under such symmetry group.

If we want to add mass terms for the lepton, however, we will end up breaking such symmetry. Now, the mass terms must still come from gauge invariant terms, and need to be the result of symmetry breaking. We need to build an $SU(2)_L \times U(1)_Y$ invariant term, and it needs to include the field L_L and the field e_R. To make it $SU(2)_L$ invariant, we need another $SU(2)_L$ doublet to contract the L_L doublet. This could be the higgs doublet Φ. However, we also need such term to be a singlet under $U(1)_Y$. This would happen only if the higgs doublet would have a specific values of Y, equal to $Y_L - E_e$. Luckily, the value of Y_Φ takes exactly this value! So we can add to the lagrangian a term

$$-y^L_{ij} \bar{L}_{L,i} e_{R,j} \Phi + h.c. \tag{1.2}$$

Unlike what we have done so far, where all terms where coming from covariant derivatives, this term is something that we add add hoc, without following the "minimal coupling" principle, but still respecting the gauge symmetry of the lagrangian. In the unitary gauge, where the goldstone bosons decouple, this will take the form

$$-y^L_{ij} \bar{e}_{L,i} e_{R,j} \frac{v+h}{\sqrt{2}} + h.c. \tag{1.3}$$

[1] Note that we are neglecting meutrino masses, given that their origin is unknown.

giving a mass matrix

$$M_{ij}^e = y_{ij}^L \frac{v}{\sqrt{2}} \tag{1.4}$$

The matrix y_{ij}^L is not necessarily an hermitian matrix, and is complex in general. It multiplies different fields on the 2 sides. To diagonalise this matrix, we need to rotate separately the left and right fields. We start by noting that

$$y_{ij}^L (y_{jl}^L)^\dagger \tag{1.5}$$

is hermitian, and can be diagonalised by a transformation $SU(3)_{L_L}$:

$$y_{ij}^L (y_{jl}^L)^\dagger = U_e D_e^2 U_e^\dagger \tag{1.6}$$

Similarly,

$$(y_{ij}^L)^\dagger y_{jl}^L \tag{1.7}$$

is also hermitian, and can be diagonalised by a transformation $SU(3)_{e_R}$:

$$(y_{ij}^L)^\dagger y_{jl}^L = W_e D_e^2 W_e^\dagger \tag{1.8}$$

Note that the diagonal matrix D_e is the same in both cases, as the eigenvalues of the 2 products are the same on both cases. If D_e has only positive eigenvalues, it can be shown that these decompositions are unique, and that

$$y^L = U_e D_e W_e^\dagger \tag{1.9}$$

The matrices $U_e, W_e^\dagger$ will get reabsorbed by the left and right flavour transformations, leaving the diagonal mass matrix

$$M^e = D_e \frac{v}{\sqrt{2}} \tag{1.10}$$

All other terms in the lagrangian are invariant separately under both transformations, so no trace of such transformation remains after it is done. So the parameters in U_e, W_e are not observable, and the yukawa lepton sector brings us 3 additional parameters only, the masses m_e, m_μ, m_τ.

The original symmetry group $U(3)_L \times U(3)_R$ is broken. If the masses were all degenerate, the group would break to $U(3)_L \times U(3)_R \to U(3)_V = U(1)_L \times SU(3)_V$ as we have seen previously. Given that the masses are not degenerate, $SU(3)_V$ breaks to the diagonal subgroup

$$U(3)_L \times U(3)_R \to U(1)_e \times U(1)_\mu \times U(1)_\tau \tag{1.11}$$

The symmetry groups $U(1)_l, l = e, \mu, \tau$ are called lepton family number conservation, that are exact in the SM. The fact that the lagrangian has an exact global symmetry under these groups implies that the number of e, μ, τ leptons, N_e, N_μ, N_τ, is separately conserved in each interaction. A subgroup of such group is $U(1)_L$, called lepton number conservation, that implies the conservation of the total lepton number $N_L = N_e + N_\mu + N_\tau$.

1.2 Yukawa quark sector

We can try to apply the same method to give mass to quarks. this time, the difference will be that both up and down quarks need to get a mass. for down quarks, once again the value of $Y_Q - Y_d = -\frac{1}{2}$ is the same as before, and we can still couple it to the higgs doublet

$$-y^d_{ij}\bar{Q}_{L,i}d_{R,j}\Phi + h.c. \tag{1.12}$$

In the case of up quarks, however, $Y_Q - Y_u = \frac{1}{2}$. We might think that we are in trouble, but luckily we can use the hermitian conjugate. Note that, to get a gauge invariant term, now the $SU(2)_L$ contraction needs to be different from usual:

$$-y^u_{ij}\bar{Q}_{L,i,a}u_{R,j}\epsilon^{ab}\Phi^\dagger_b + h.c. \tag{1.13}$$

where we made explicit the $SU(2)_L$ indices a, b.

We can operate in the same way as before, obtaining the decompositions

$$y^u = U_u D_u W_u^\dagger \tag{1.14}$$

$$y^d = U_d D_d W_d^\dagger \tag{1.15}$$

There is only one problem. In the case of quarks, the symmetry group is

$$U(3)_Q \times U(3)_u \times U(3)_d \tag{1.16}$$

So, while we are free to operate separately different rotations W_u, W_d on the right handed fields u_R, d_R, and keeping the rest of the lagrangian invariant, operating different transformations U_u, U_d on the 2 components of the left handed fields will affect some terms of the other parts of the lagrangian. We need to work in the mass eigenstats basis, so we will proceed anyway. We get the mass terms as with leptons:

$$M^u = D_u\frac{v}{\sqrt{2}} \tag{1.17}$$

$$M^d = D_d\frac{v}{\sqrt{2}} \tag{1.18}$$

By applying the U_u, U_d transformation on left handed fields, we will get a factor

$$V_{CKM} = U_u^\dagger U_d \tag{1.19}$$

in any term of the lagrangian that was contracting a left handed u-family quark with a left handed d-family quark. There is only one such term in the lagranian, the one giving the interaction with the W boson. This allows the weak interaction to change a quark of a family in a quark of any other family. Note that while the W interaction vertex now turns into

$$-i\frac{g}{\sqrt{2}}\gamma^\mu P_L\delta^{ij} \to -i\frac{g}{\sqrt{2}}\gamma^\mu P_L V^{ij} \tag{1.20}$$

thus connecting all up quarks with all down quarks, the interaction vertex with both the photon and the Z boson remains flavour-diagonal, as they connect only up quark with up quarks, and down quarks with down quarks. At tree level, flavour is conserved in SM neutral currents. At loop level, we will see that, thanks to the GIM mechanism.

Of the original symmetry group, only the subgroup $U(1)_B$ of baryon number conservation survives. One can check, by counting d.o.f., that for n_F families, the CKM matrix has

$$\frac{n_F(N_F-1)}{2} \tag{1.21}$$

real parameters (angles), and

$$\frac{(n_F-1)(n_F-2)}{2} \tag{1.22}$$

phases. For $n_F = 3$ we obtain 3 angles and 1 phase. The existence of a non-zero phase is a source of CP violation in the standard model.

The CKM matrix elements are approximately

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{1.23}$$

1.3 GIM Mechanism

The GIM mechanism suppressed FCNC at loop level, combining the unitarity of the CKM matrix with the smallness of quark masses. For example, Calculate the Amplitude $b \to s\gamma$. The amplitude will include an external b quark leg, and external s quark leg, and one or more internal lines with a quark of the up family. It will also include 2 vertices with the W boson, that will bring a factor $V_{bl}V^\dagger_{ls}$. For a given internal line flavour l, the amplitude will be

$$\mathcal{M}_l = V_{bl}V^\dagger_{ls}F\left(\frac{m_l^2}{M_W^2}\right) \tag{1.24}$$

Now, assuming $m_l \ll M_W$, the function can be expanded in power of $\frac{m_l^2}{M_W^2}$, and the result is

$$\sum_l \mathcal{M}_l = \sum_l V_{bl}V^\dagger_{ks}\left(A\delta_{lk} + BD^2_{u,lm} + \mathcal{O}(D_u^4)\right) \tag{1.25}$$

$$= \sum_l V_{bl}V^\dagger_{ks}\left(BD^2_{u,lm} + \mathcal{O}(D_u^4)\right) \tag{1.26}$$

where the first term cancels out due to the unitarity of the matrix V. For light quarks, the $\frac{m_l^2}{M_W^2}$ factor will suppress the contribution, while in the case of the top quark the contribution will still be suppressed thanks to the smallness of V_{13}, V_{23}.

References