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The Standard Model Lagrangian (Lecture 3)

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Abstract.

Contents

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1.1 Higgs mechanism in the SM: EW symmetry breaking

So far we have built the Standard Model as a chiral Yang-Mills theory, including the gauge bosons from the relative gauge groups, and 5 chiral fermionic fields, and all particles are massless. The only parameters that have appeared in the theory so far are the 3 Sm gauge couplings, $g_{1,2,3}$, or alternatively $g, g^{'}, g_s$. There are a few problems however: first, we know that all fermions are not massless, but rather have a certain nonzero mass. Second, while the photon and gluons are observed to be massless, that means their gauge group is unbroken, the W, Z bosons are observed to be massive, with a mass of $81, 90 GeV$ respectively. Moreover, the photon is the gauge boson associated to $U(1)_{em}$, a non-chiral subgroup of the SM gauge group, while the SM gauge group seems to offer a different $U(1)$ subgroup, the $U(1)_Y$ subgroup, that is chiral (Y are not the same for left and right handed fermions).

An explicit mass term for W, Z bosons is not allowed, and would break gauge symmetry. The Z boson so far does not even appear explicitly in the lagrangian, as we have $W^{1,2,3}$. Moreover, as the theory is chiral, also mass terms for fermions are all explicitly forbidden by gauge invariance: we cannot just them to the lagrangian by hand.

To give masses to the W, Z bosons, the only possible way to to spontaneously break the gauge symmetry. This is the mechanism invented by Peter Higgs. Under certain conditions, the same particle can be used to give masses to the fermions of the theory. We will see that indeed such conditions are met in the SM, and so we can solve all the problems at once.

To give masses to W, Z, we will need to break 3 generators, once for each d.o.f.: W^+, W^-, Z. From the Goldstone theorem, we would expect that for each broken generator we will get a massless goldstone boson. We will see that as we are breaking a local (gauged) symmetry this time thigns will be a little different: the scalar will not become massless, but will rather get "eaten" by the gauge boson to become its longitudinal polarization d.o.f.. In any case, we know that we will need to have, before the breaking, at least one scalar for each generator we want to break.

Moreover, we understand that we need to break down the $SU(2)_L \times U(1)_Y$ subgroup to $U(1)_{em}$ subgroup, and not just break the $SU(2)_L$ part itsself, for 2 reasons. First, in such case one would be left, as (1) remaining subgroup, that $U(1)_Y$, that is chiral and cannot be associated to the photon. Second, by breaking the $SU(2)_L$ part alone, one would get 3 gauge bosons with identical masses, while we know that $M_Z \neq M_W$.

To achive such breaking pattern, we need to add to the theory a scalar in a representation charged under both $SU(2)_L$ and $U(1)_Y$. Such representation should contain at least 3 real scalar particles, to heave at least one scalar for each generator we want to break. It is easy to check that the very minimal choice of an $SU(2)_L$ doublet with nonzero Y satisfies all such requirements, as it contains 2 complex scalars, so 4 real scalars d.o.f.. We will call such doublet Φ. The 2 components of Φ will have an electric charge given by the relation

$$Q = T_3 + Y \tag{1.1}$$

We want one of the components to be of neutral charge, so that such component can develop a nonzero vev. We cannot assign a non-zero vev to a charged component, as this would generate a breaking pattern that would not leave the $U(1)_{em}$ subgroup unbroken. In other words, the photon would get a mass, and we would get a different massless gauge boson in the theory. As $T_3 = \pm\frac{1}{2}$, we need either $Y = \mp\frac{1}{2}$. We choose $Y = \frac{1}{2}$. In this case, the $T_3 = -\frac{1}{2}$ will be charge neutral, and can develop a nonzero vev v:

$$\langle\Phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \tag{1.2}$$

The gauge transformation law for Φ will be

$$\Phi \to e^{i\alpha^a t^a} e^{iY\alpha^0}\Phi \tag{1.3}$$

Setting $Y = \frac{1}{2}$, we can check explicitly that the gauge transformation with $\alpha^1 = \alpha^2 = 0, \alpha^3 = \alpha^0$ leaves $\langle\Phi\rangle$ invariant. This guarantees that there will be a masseless gauge boson remaining in the theory, associated with the electric charge.

1.1.1 Gauge Boson masses

To find the masses of the EW gauge bosons in terms of v, all we need to do is to write the kinetic term for the new field and isolate the mass terms coming from the covariant derivative:

$$\begin{aligned}
D_\mu &= \partial_\mu - igW_\mu^a t^a - ig^{'}Y_\Phi B_\mu && (1.4)\\
&= \partial_\mu - igW_\mu^a t^a - ig^{'}\frac{1}{2}B_\mu && (1.5)\\
&= \begin{pmatrix}\partial_\mu - i\frac{1}{2}gW_\mu^3 - i\frac{1}{2}g^{'}B_\mu & -i\frac{1}{2}g(W_\mu^1 - iW_\mu^2)\\ -i\frac{1}{2}g(W_\mu^1 + iW_\mu^2) & \partial_\mu + i\frac{1}{2}gW_\mu^3 - i\frac{1}{2}g^{'}B_\mu\end{pmatrix} && (1.6)\\
\mathcal{L}_{kin} &= D_\mu\Phi D^\mu\Phi^\dagger && (1.7)
\end{aligned}$$

We can group the $W^{1,2}$ real fields into complex fields with ±1 electric charge as

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right) \tag{1.8}$$

We get

$$D_\mu = \begin{pmatrix}\partial_\mu - i\frac{1}{2}gW_\mu^3 - i\frac{1}{2}g^{'}B_\mu & -i\frac{1}{\sqrt{2}}gW_\mu^+\\ -i\frac{1}{\sqrt{2}}gW_\mu^- & \partial_\mu + i\frac{1}{2}gW_\mu^3 - i\frac{1}{2}g^{'}B_\mu\end{pmatrix} \tag{1.9}$$

$$\Phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix} \tag{1.10}$$

Now, the mass terms will need to include 2 powers of the gauge fields W, B, and a coefficient of dimensions of squared energy, so they cannot contain derivatives. So we can drop the derivative part. We can also drop all fields inside Φ, as they will not appear in the mass terms for the gauge bosons, so

$$\mathcal{L}_{W,Zmass} = \left| \begin{pmatrix} -i\frac{1}{2}gW^3_\mu - i\frac{1}{2}g^{'}B_\mu & -i\frac{1}{\sqrt{2}}gW^+_\mu \\ -i\frac{1}{\sqrt{2}}gW^-_\mu & +i\frac{1}{2}gW^3_\mu - i\frac{1}{2}g^{'}B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}v \end{pmatrix} \right|^2 \tag{1.11}$$

$$= \begin{pmatrix} -i\frac{gv}{2}W^+_\mu \\ -i\frac{v}{2\sqrt{2}}\left(g^{'}B_\mu - gW^3_\mu\right) \end{pmatrix} \begin{pmatrix} i\frac{gv}{2}W^-_\mu \\ i\frac{v}{2\sqrt{2}}\left(g^{'}B_\mu - gW^3_\mu\right) \end{pmatrix} \tag{1.12}$$

$$= \begin{pmatrix} -i\frac{gv}{2}W^+_\mu \\ -i\frac{v\sqrt{g^2+(g^{'})^2}}{2\sqrt{2}}\frac{g^{'}B_\mu - gW^3_\mu}{\sqrt{g^2+(g^{'})^2}} \end{pmatrix} \begin{pmatrix} i\frac{gv}{2}W^-_\mu \\ i\frac{v\sqrt{g^2+(g^{'})^2}}{2\sqrt{2}}\frac{g^{'}B_\mu - gW^3_\mu}{\sqrt{g^2+(g^{'})^2}} \end{pmatrix} \tag{1.13}$$

We can identify already the properly normalised linear combination of W^3, B that will make up the massive Z boson as:

$$Z_\mu = \frac{g^{'}B_\mu - gW^3_\mu}{\sqrt{g^2 + (g^{'})^2}} \tag{1.14}$$

The orthogonal component, the photon, will instead remain massless

$$A_\mu = \frac{gB_\mu + g^{'}W^3_\mu}{\sqrt{g^2 + (g^{'})^2}} \tag{1.15}$$

When one has multiple fields, it might not be as easy to identify the mass eigenstates from the expression. One therefore writes the "mass matrix", for the fields a, b, as

$$M_{ab} = \frac{\partial^2 \mathcal{L}}{\partial a \partial b^\dagger} \tag{1.16}$$

Such mass matrix is always block diagonal, with one block for each possible value of the electric charge. Neutral scalars, if the CP symmetry is conserved, are also separated into 2 different blocks, one for CP even ($CP = +$) and one for CP odd ($CP = -$). We can indeed check that, for $a = W^+, W^3, B$ our mass matrix is

$$\begin{pmatrix} \frac{g^2v^2}{4} & 0 & 0 \\ 0 & \frac{g^2v^2}{4} & -\frac{gg^{'}v^2}{4} \\ 0 & -\frac{gg^{'}v^2}{4} & \frac{(g^{'})^2v^2}{4} \end{pmatrix} \tag{1.17}$$

The first column and row refer to a particle of charge $Q + \pm 1$, and is indeed block diagonal. The remaining 2×2 block can be diagonalised, with the eigenvalues giving the values of the masses, and the relative (normalised) eigenvectors giving the right linear combinations that generate the mass eigenstates. Note that the definition of the mass matrix automatically account for the $1/2$ factor in the case of real fields. The diagonalised matrix becomes

$$\begin{pmatrix} \frac{g^2v^2}{4} & 0 & 0 \\ 0 & \frac{(g^2+(g^{'})^2)v^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.18}$$

From the mass matrix we can immediately read the mass values:

$$M_W = \frac{g}{2}v \tag{1.19}$$

$$M_Z = \frac{\sqrt{g^2+(g^{'})^2}}{2}v \tag{1.20}$$

$$M_A = 0 \tag{1.21}$$

Substituting back into the lagragian the expressions for B, W^3, one can obtain the couplings of the fermions to the Z, A gauge bosons. Moreover, the other terms that we have discarded coming from the kinetic term of the Higgs field will also generate interactions between the gauge fields and the higghs field h. We will also get interaction between gauge fields, higgs field and the goldstone bosons $G^{\pm}, G^0$. We will need to understand what those term mean and if they are physical. We can rewrite the covariant derivative for a generic field charged under $SU(2)_L \times U(1)_Y$ in terms of the mass eigenstates as

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ t^+ W_\mu^- t^-\right) - i\frac{g^2T^3-(g^{'})^2Y}{\sqrt{g^2+(g^{'})^2}}Z_\mu - i\frac{gg^{'}\,(T_3+Y)}{\sqrt{g^2+(g^{'})^2}}A_\mu \tag{1.22}$$

$$= \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ t^+ W_\mu^- t^-\right) - i\frac{g^2T^3-(g^{'})^2Y}{\sqrt{g^2+(g^{'})^2}}Z_\mu - ieQA_\mu \tag{1.23}$$

where we have identified

$$Q = T^3 + Y \tag{1.24}$$

$$e = \frac{gg^{'}}{\sqrt{g^2+(g^{'})^2}} \tag{1.25}$$

To change the base from the "flavour" base to the mass eigenstate base, it is convenient to define the angle θ_w:

$$\begin{pmatrix} Z \\ A \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix} \tag{1.26}$$

with

$$\cos\theta_w = \frac{g}{\sqrt{g^2+(g^{'})^2}} = \frac{M_W}{M_Z} \tag{1.27}$$

$$\sin\theta_w = \frac{g^{'}}{\sqrt{g^2+(g^{'})^2}} \tag{1.28}$$

the coupling to Z can be rewritten in terms of Q rather than Y:

$$g^2T^3 - (g^{'})^2Y = (g^2+(g^{'})^2)T^3 - (g^{'})^2Q \tag{1.29}$$

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+ t^+ W_\mu^- t^-\right) - i\frac{g}{\cos\theta_w}(T^3 - \sin^2\theta_w Q)Z_\mu - ieQA_\mu \tag{1.30}$$

with

$$e = \frac{g}{\sin\theta_w} \tag{1.31}$$

by taking the low energy limit, we can connect g, M_W with the fermi constant

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{1.32}$$

1.1.2 The origin of the higgs vev

In order to develop a nonzero vev, we need to write down the potential for the scalar field and make it such that the minimum is located at a nonzero value of the field. Due to gauge invariance, the only gauge invariant combination that can appear in the lagrangian is

$$\Phi\Phi^\dagger \tag{1.33}$$

So the most general potential will be

$$V(\Phi) = \mu^2\Phi\Phi^\dagger + \lambda(\Phi\Phi^\dagger)^2 \tag{1.34}$$

If $\mu^2 > 0$ the potential has a minimum for $\Phi = 0$ and develops no vev: the theory remains unbroken. So we need to change the sign of μ^2. As we like to keep μ^2 positive, let's add a $-$ sign in front

$$V(\Phi) = -\mu^2\Phi\Phi^\dagger + \lambda(\Phi\Phi^\dagger)^2 \tag{1.35}$$

Now the potential develops a minimum for a nonzero value of Φ. Derivating w.r.t Φ we get

$$\frac{\partial V}{\partial \Phi} = \Phi^\dagger\left(-\mu^2 + 2\lambda\Phi\Phi^\dagger\right) \tag{1.36}$$

The minimum therefore has to satisfy

$$\frac{\partial V}{\partial \Phi}(\Phi_0) = \Phi_0^\dagger\left(-\mu^2 + 2\lambda\Phi_0\Phi_0^\dagger\right) = 0 \tag{1.37}$$

$$-\mu^2 + 2\lambda\Phi_0\Phi_0^\dagger = 0 \tag{1.38}$$

$$\Phi_0\Phi_0^\dagger = \frac{\mu^2}{2\lambda} \tag{1.39}$$

The combination $\Phi_0\Phi_0^\dagger$ is invariant under gauge transformations. However, we have taken a well defined value in the previous section for Φ_0:

$$\Phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{1.40}$$

so we get

$$v^2 = \frac{\mu^2}{\lambda} \tag{1.41}$$

We now want to find the higgs mass. We may also want to check the masses of the Goldstone particles $G^\pm, G^0$. For this, we can, as usual, get the mass matrix $(a, b = (h, G^0, G^+))$

$$M_{ab} = \frac{\partial^2 V}{\partial a \partial b^\dagger}|_{\Phi=\Phi_0} = \begin{pmatrix} 2\lambda v^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.42}$$

So we indeed confirm that all goldstone bosons are massless, and that

$$m_h^2 = 2\lambda v^2 \tag{1.43}$$

1.2 Feynman rules for Spontaneously broken Gauge theories: ξ gauges

Now that we saw how to break the theory at the classical level, we want to understand how to quantise it, and in particular what our propagators are. We also want to understand if the goldstone bosons are physical of not. Such massless scalars are indeed not observed in nature, so we suspect that they are unphysical.

Going back to the kinetic-gauge terms coming form the covariant derivative of the Higgs fields, if we expand all terms, we notice that there are some additional terms that are quadratic in the fields and that are not "standard":

$$\mathcal{L} = D_\mu \Phi D^\mu \Phi^\dagger \tag{1.44}$$

If we extract all terms that are quadratic in the scalar and vector fields, we get

$$\begin{aligned}\mathcal{L}_{quadratic} = & \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu G^0 \partial^\mu G^0 + \partial_\mu G^+ \partial^\mu G^- + \frac{1}{2}M_Z^2 Z_\mu Z^\mu + M_W^2 W_\mu^+ W^{-,\mu} \\ & -\frac{1}{2}\sqrt{g^2 + (g')^2} v Z^\mu \partial_\mu G^0 - \frac{1}{2} g v W^{+,\mu}\partial_\mu G^- - \frac{1}{2} g v W^{-,\mu}\partial_\mu G^+ \end{aligned} \tag{1.45}$$

While we were expecting the terms of the first line, the ones on the second line are unexpected and problematic. We need to implement, as usual the "Gauge fixing" to get rid of them. The usual gauge fixing term we used was

$$\mathcal{L}_{GF} = -\frac{1}{2}(G^a)^2 - \bar{c}^a \partial^\mu D_\mu^{ac} c^c = -\frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)^2 - \bar{c}^a \partial^\mu D_\mu^{ac} c^c \tag{1.46}$$

Using functional integration, one can check that one needs to modify G and the ghost terms as follows:

$$G^A = \partial^\mu A_\mu \tag{1.47}$$

$$G^Z = \partial^\mu Z_\mu - \xi M_Z G^0 \tag{1.48}$$

$$G^W = \partial_\mu W_\mu^+ - \xi M_W G^+ \tag{1.49}$$

So the G term, together with the mass and kinetic terms, brings the quadratic part of the lagrangian in the form

$$\mathcal{L} = -\frac{1}{2} A_\mu^a \left((-g^{\mu\nu}\partial^2 + (1 - \frac{1}{\xi})\partial^\mu \partial^\nu)\delta^{ab} - M^{ab} \right) A_\nu^b + \frac{1}{2} G^i \left(-\partial^2 - \xi M^{ij}\right) G^j \tag{1.50}$$

this gives us the propagators of the theory. For the scalars, the gauge fixing term gives them a mass

$$M_{G^0}^2 = \xi M_Z^2 \tag{1.51}$$

$$M_{G^\pm}^2 = \xi M_W^2 \tag{1.52}$$

As the mass is gauge-dependent, it is clear that such particles must be unphysical. However, for a generic gauge ξ we must include the feynman diagrams that include such unphysical particles as internal lines, as much as we do for ghosts, to retain gauge invariance. One special exception is, at tree level, the limit $\xi \to \infty$. In this limit, the particles become infinitely massive and decouple from the theory, as there is no vertex that is $\propto \xi$. This is called the unitary gauge. With a bit of work, we can also get the propagator for the W, Z fields

$$\frac{-i}{q^2 - M^2}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2 - \xi M^2}\right) \tag{1.53}$$

F	$SU(3)$	$SU(2)$	$U(1)$
$\bar{Q}_L$	3	2	$-1/6$
u_R	3	1	$2/3$
d_R	3	1	$-1/3$
$\bar{L}_L$	1	2	$1/2$
e_R	1	1	-1

Table 1: Standard Model Fermion field content

In the unitary gauge, this simplifies to

$$\frac{-i}{q^2 - M^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M^2}\right) \tag{1.54}$$

Similarly, the sums over the external polarizations will now take the form

$$\sum \epsilon^\mu \epsilon^{\nu,*} = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M^2}\right) \tag{1.55}$$

The ghost term gets modified as well. Ghosts acquire a mass squared ξM^2, so their propagator reads,

$$\frac{i}{q^2 - \xi M^2} \tag{1.56}$$

and also get interactions with physical higgs and goldstone bosons. We will not go into the details, but the ghost lagrangian can be obtained as usual from the gauge variation of G^a.

1.3 Anomaly cancellation in the SM gauge group

Any proper gauge theory needs the relative gauge group to be not anomalous, as anomalies would spoil gauge invariance.

It can be shown that the anomalies are proportional to the gauge group invariant

$$Tr[\gamma^5 t^a \{t^b, t^c\}] \tag{1.57}$$

with the sum takes over all fermion species/representations. The γ^5 factor reminds the fact that anomalies are associated to chiral symmetries, with a factor $+1$ for tight handed fermions and -1 for left handed fermions. Any non-chiral theory, like QED and QCD where right and left handed fermions couple equally to gauge bosons have their anomalies automatically cancelled. For convenience, we can take all fermions to be right handed, by replacing the particles with the relative antiparticles for any left handed fermion representation. This will flip all $U(1)_Y$ charges in tab. **??**. We obtain Tab. 1.

There are 10 possible combination, however most of them automatically cancel.

1. $SU(3)^3$ cancels as QCD is not chiral
2. $SU(2)^3$ cancels as it is a special property of $SU(2)$

3. Any combination including either a single $SU(3)$ of $SU(2)$ factor will be proportional to the trace of a single generator, that always vanishes.

This leaves only 3 non-trivial factors to check. The first one is $SU(3)^2 \times U(1)$:

$$\frac{1}{2}\left(2(-\frac{1}{6})+\frac{2}{3}-\frac{1}{3}\right)=0 \tag{1.58}$$

The second factor is $SU(2)^2 \times U(1)$:

$$\frac{1}{2}\left(3(-\frac{1}{6})+\frac{1}{2}\right)=0 \tag{1.59}$$

Finally, the last factor to check out is $U(1)^3$

$$6(-\frac{1}{6})^3+3(\frac{2}{3})^3+3(-\frac{1}{3})^3+2(\frac{1}{2})^3+(-1)^3=-\frac{1}{36}+\frac{8}{9}-\frac{1}{9}+\frac{1}{4}-1=0 \tag{1.60}$$

The $U(1)_Y$ charges are related to the EM charges by

$$Q=T_3+Y \tag{1.61}$$

Note that there is also another definition of the hypercharge, where $Y \to \frac{1}{2}Y$. Using this equation, together with the fact that the EM charge for the various right and left handed particle are the same (meaning that QED preserves parity), gives us 4 linear equations, that sum up to the 3 equations (2 linear, 1 non-linear) from anomaly cancellation. Looking at the linear equations, one of them is linearly dependent form the other, so this leaves us with 5 linear equations and one non-linear equation, for 5 unknown charges. The 5 linear equations alone would yield an unique solution, so it can seem to be a miracle that also the additional non-linear equation is satisfied. This cancellation appears to be magic: indeed we will see that one can predict these cancellations based on the initial representations and a bit of group theory.

References