January 11, 2023

The Standard Model Lagrangian (Lecture 2)

Giorgio Busoni[a]

[a]Department of Nuclear Physics, Research School of Physical Sciences and Engineering, Australian National University, Canberra, Australia

E-mail: giorgio.busoni@anu.edu.au

Abstract.

Contents

1 Optical Theorem and Unitarity

1.1 The Optical theorem and decay widths

The $\mathbb{S}$ matrix can be written as

$$\mathbb{S} = \mathbb{1} + i\mathbb{T} \tag{1.1}$$

This matrix is unitary, so

$$\mathbb{S}\mathbb{S}^\dagger = (\mathbb{1} + i\mathbb{T})(\mathbb{1} - i\mathbb{T}^\dagger) = \mathbb{1} \tag{1.2}$$

$$0 = i(\mathbb{T}_{ij} - \mathbb{T}^\dagger_{ij}) + \mathbb{T}_{il}\mathbb{T}_{lj} \tag{1.3}$$

The $\mathbb{T}$ matrix is infinite dimensional, with each row for each set of particles and for each momentum k^μ. We can relate this to the $\mathcal{M}$ matrix by

$$\mathbb{T}_{ij} = \mathcal{M}_{ij}(2\pi)^4\delta^4(\sum p_i - \sum p_j) \tag{1.4}$$

and replacing the sum over the index l by

$$\sum_l \to \sum_l \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \tag{1.5}$$

We get

$$-i(\mathcal{M}_{ij} - M^\dagger_{ji})\delta^4(p_i - p_j) = \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{lj}\delta^4(p_i - p_l)(2\pi)^4\delta^4(p_l - p_j) \tag{1.6}$$

$$= \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{lj}\delta^4(p_i - p_j)(2\pi)^4\delta^4(p_l - p_j) \tag{1.7}$$

and so

$$(-i(\mathcal{M}_{ij} - M^\dagger_{ji}) - \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{lj}(2\pi)^4\delta^4(p_l - p_j))\delta^4(p_i - p_j) = 0 \tag{1.8}$$

We now choose the case $i = j$. The overall delta function is not zero, so what is in the brackets must be zero. We get

$$-i(\mathcal{M}_{ii} - M^\dagger_{ii}) - = \prod_{p_l} \frac{d^3p_l}{(2\pi)^3 2E_{p_l}} \mathcal{M}_{il}\mathcal{M}^\dagger_{li}(2\pi)^4\delta^4(p_l - p_j) \tag{1.9}$$

$$2Im[\mathcal{M}_{ii}] = |\mathcal{M}_{il}|^2 d\Phi_{LIPS} \tag{1.10}$$

where the sum over all possible l states is assumed.

Now we remind the equation to calculate a cross section

$$d\sigma = \frac{1}{2E_1 2E_2|v_1 - v_2|}|\bar{\mathcal{M}}|^2 d\Phi_{LIPS} \tag{1.11}$$

The equation for a (partial) decay width (inverse of decay time) is

$$d\Gamma = \frac{1}{2M}|\bar{\mathcal{M}}|^2 d\Phi_{LIPS} \tag{1.12}$$

We get

$$Im[\mathcal{M}_{ii}] = M\Gamma \tag{1.13}$$

Where we have "assumed" $|\bar{\mathcal{M}}|^2 = |\mathcal{M}|^2$. This is true for scalar particles. We will see that this relation holds also for other particles. Such particles will have spinor or Lorentz indices in their self energy, and correctly accounting for them will keep such relation true.

The optical theorem tells us that the matrix elements of the $\mathcal{M}$ matrix are related to each other due to the unitarity constraint of the $\mathbb{S}$ matrix. Our first conclusion, Eq. 1.13, tells us that the theory is not complete at loop level: in fact, while one can usually calculate the decay amplitude at loop level, the matrix element $\mathcal{M}_{ii}$ does not develop an imaginary part at tree level (at tree level one has just the propagator). We will see that the decay width is not just a property of a particle that tells us about its lifetime, but that it can play important roles in certain other situations. But, before that, we will now see a few examples of application of the optical theorem to self energies and decay widths.

We now go through some examples.

1.2 Unitarity limit on cross sections from Optical Theorem

Starting back from the equation

$$2Im[M_{ii}] = \int |M_{ij}|^2 d\Phi_{LIPS} \tag{1.14}$$

We can specialise now to the case of a 2 particle initial and final state. For simplicity, we also assume particles of the same mass, but the result will be general

$$2Im[M_{ii}] = \frac{1}{8\pi}\sqrt{1-\frac{4m_j^2}{s}}|M_{ij}|^2 \tag{1.15}$$

We can now place an upper bound on the LHS and a lower bound on the RHS

$$2|M_{ii}| \geq 2Im[M_{ii}] = \sum_j \frac{1}{8\pi}\sqrt{1-\frac{4m_j^2}{s}}|M_{ij}|^2 \geq \frac{1}{8\pi}\sqrt{1-\frac{4m_i^2}{s}}|M_{ii}|^2 \tag{1.16}$$

We get the inequality

$$2|M_{ii}| \geq \frac{1}{8\pi}\sqrt{1-\frac{4m_i^2}{s}}|M_{ii}|^2 = \frac{v}{8\pi}|M_{ii}|^2 \tag{1.17}$$

$$|M_{ii}| \leq \frac{16\pi}{v} \tag{1.18}$$

We obtain a bound on the elastic cross section $i \to i$ (we use the c.o.m. frame)

$$\sigma = \frac{1}{2E_1 2E_2|v_1 - v_2|}|M_{ii}|^2\frac{v}{8\pi} \tag{1.19}$$

$$= \frac{1}{8E^2 v}|M_{ii}|^2\frac{v}{8\pi} \tag{1.20}$$

$$\leq \frac{1}{64\pi E^2}\left(\frac{16\pi}{v}\right)^2 = \frac{4\pi}{E^2 v^2} = \frac{4\pi}{p_{com}^2} \to_{s\to\infty} \frac{16\pi}{s} \tag{1.21}$$

We can verify this bound using our scalar example, but this time let's assume that the particle S decays into 2 distinguishable particles f the same mass, so that we don't have the 1/2 factor in the phase space. The width of S is

$$\Gamma = \frac{\mu^2}{16\pi M}\sqrt{1-\frac{4m^2}{M^2}} \tag{1.22}$$

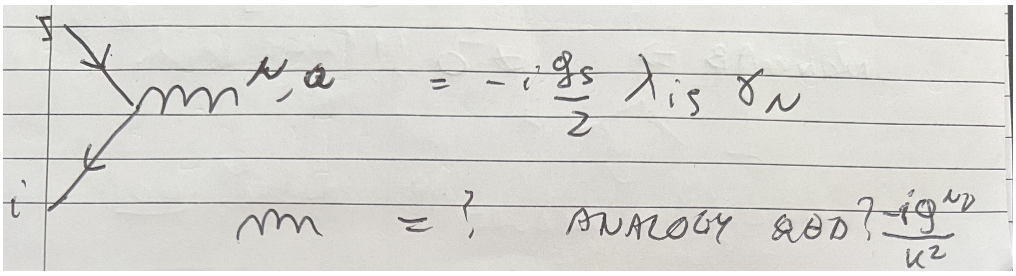

Figure 1: Gluon Feyman rules

The dressed propagator for S becomes

$$\frac{i}{Q^2 - M^2 - iM\Gamma} \tag{1.23}$$

The matrix element for $\eta\eta' \to \eta\eta'$ is

$$i\mathcal{M} = \frac{i(-i\mu)^2}{s - M^2 - iM\Gamma} \tag{1.24}$$

$$|\mathcal{M}|^2 = \frac{\mu^4}{(s - M^2)^2 + M^2\Gamma^2} \tag{1.25}$$

$$\sigma(s) = \frac{1}{2sv}\frac{v}{8\pi}|\mathcal{M}|^2 = \frac{1}{16\pi s}\frac{\mu^4}{(s - M^2)^2 + M^2\Gamma^2} \tag{1.26}$$

The peak of the cross section is reached at the resonance, for $s = M^2$. For such energy we get

$$\sigma(M^2) = \frac{\mu^4}{16\pi M^4\Gamma^2} = \frac{\mu^4}{16\pi M^4 \frac{\mu^4}{(16\pi)^2 M^2} v^2} \tag{1.27}$$

$$= \frac{1}{M^2 \frac{1}{(16\pi)} v^2} = \frac{16\pi}{sv^2} = \frac{4\pi}{p_{com}^2} \tag{1.28}$$

2 Gauge Invariance for Non-Abelian Gauge groups

2.1 Ward Identity

The feynman rules for the fermion-gluon vertex can be worked out easily, also in analogy with QED.

$$-ig_s\gamma^\mu(t^a)_{ij} \tag{2.1}$$

However, one may wonder what is the correct feynman rule for the gluon propagator (see Fig.1). One could be tempted to use

$$\frac{-ig_{\mu\nu}}{k^2}. \tag{2.2}$$

However, we have to worry about the ward identity to be satisfied. To check if it is, we will now compute it for the following process

$$q\bar{q} \to gg. \tag{2.3}$$

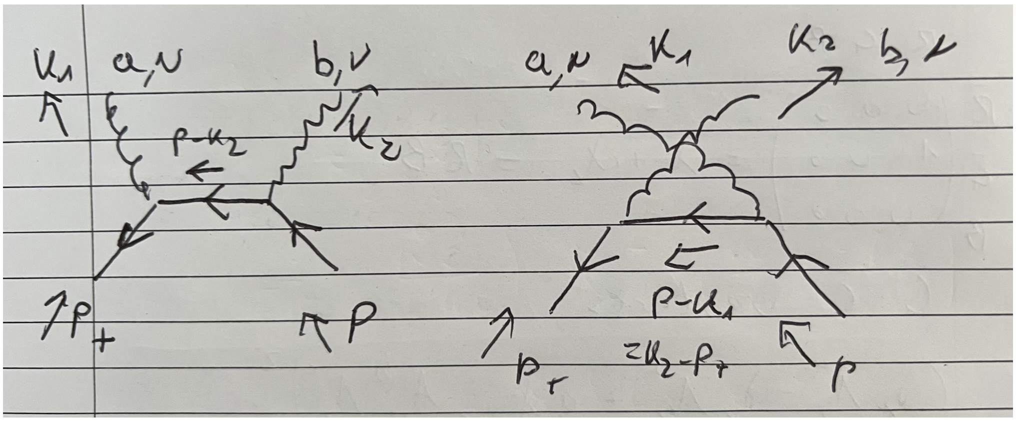

Figure 2: QED-like feyman diagrams for $q\bar{q} \to gg$

We start by calculating the QED-like diagrams gives in Fig. 2.

Factorising the external polarizations and contracting one index with the relative gluon momenta, we get

$$iM^{\mu\nu}_{1+2}k_{2\nu} = (ig_s)^2 \bar{v}(p_+) \left(\gamma^\mu t^a \frac{i}{\slashed{p} - \slashed{k}_2 - m} \slashed{k}_2 t^b + \slashed{k}_2 t^b \frac{i}{\slashed{k}_2 - \slashed{p}_+ - m} \gamma^\mu t^a \right) u(p) \tag{2.4}$$

By using

$$(\slashed{p} - m)u(p) = 0, \quad \bar{v}(p_+)(-\slashed{p}_+ - m) = 0, \tag{2.5}$$

we get

$$\begin{aligned} iM^{\mu\nu}_{1+2}k_{2\nu} &= (ig_s)^2 \bar{v}(p_+) \left(\gamma^\mu t^a \frac{i(\slashed{k}_2 - (\slashed{p} - m))}{\slashed{p} - \slashed{k}_2 - m} t^b + t^b \frac{i(\slashed{k}_2 - \slashed{p}_+ - m)}{\slashed{k}_2 - \slashed{p}_+ - m} \gamma^\mu t^a \right) u(p) && (2.6) \\ &= i(g_s)^2 \bar{v}(p_+) \gamma^\mu u(p) [t^a, t^b] && (2.7) \end{aligned}$$

For QED, this is zero as the group is abelian. For non-abelian group, this is non-zero. This is ok, as there is a third diagram in the case of non-abelian gauge group:

$$iM^{\mu\nu}_3 k_{2\nu} = ig_s \bar{v}(p_+) \gamma_\rho t^c u(p) \frac{-i}{k_3^2} g_s f_{abc} \left(g^{\mu\nu}(k_2 - k_1)^\rho + g^{\nu\rho}(k_3 - k_2)^\mu + g^{\rho\mu}(k_1 - k_3)^\nu \right) k_{2\nu} \tag{2.8}$$

Now as

$$k_3 = -k_1 - k_2 \tag{2.9}$$

the part in brackets, contracted with k_2, becomes

$$g^{\rho\mu} k_3^2 - k_3^\rho k_3^\mu - g^{\rho\mu} k_1^2 + k_1^\rho k_1^\mu. \tag{2.10}$$

Using the dirac equation we note that

$$\bar{v}(p_+)\not{k}_3 u(p) = -\bar{v}(p_+)(\not{p} + \not{p}_+)u(p) = -\bar{v}(p_+)(m - m)u(p) = 0 \tag{2.11}$$

So the second term will vanish after we contract it with the gamma matrix in the fermion line. The first term instead cancels with the other 2 diagrams

$$i(M^{\mu\nu}_{1+2} + M^{\mu\nu}_3)k_{2\nu} = g_s^2 f_{abc}\bar{v}(p_+)\left(-\gamma^\mu t^c + \frac{\gamma^\mu k_3^2 - \gamma^\mu k_1^2 + \not{k}_1 k_1^\mu}{k_3^2} t^c\right) u(p) \tag{2.12}$$

$$= g_s^2 f_{abc}\bar{v}(p_+)\left(\frac{-\gamma^\mu k_1^2 + \not{k}_1 k_1^\mu}{k_3^2} t_c\right) u(p) \tag{2.13}$$

If the gluons are on shell, the first therm is zero due to $k_1^2 = 0$, and the second term cancels when contracted with the external polarization.

However, if gluons are NOT on shell, the ward identity is NOT satisfied. This is a problem for virtual particles (i.e. loop diagrams) and a disaster for renormalization!

2.2 Optical Theorem for an SU(N) theory

We can write the polarizations as

$$\xi_{T,1} = (0, 1, 0, 0) \tag{2.14}$$

$$\xi_{T,2} = (0, 0, 1, 0) \tag{2.15}$$

$$\xi_+ = (\frac{k_0}{\sqrt{2}|k_0|}, 0, 0, \frac{k}{\sqrt{2}|k|}) \tag{2.16}$$

$$\xi_- = (\frac{k_0}{\sqrt{2}|k_0|}, 0, 0, -\frac{k}{\sqrt{2}|k|}) \tag{2.17}$$

we get the following relations

$$\xi_{T,i,\mu}\xi^\mu_{T,j} = -\delta_{ij} \tag{2.18}$$

$$\xi_{\pm\mu}\xi^\mu_{T,i} = 0 \tag{2.19}$$

$$\xi_{+\mu}\xi^{+\mu} = \xi_{-\mu}\xi^{-\mu} = 0 \tag{2.20}$$

$$\xi_{+\mu}\xi^{-\mu} = 1 \tag{2.21}$$

we can rewrite the polarization sum as

$$-g_{\mu\nu} = \sum_{i,j} \xi_{T,i,\mu}\xi_{T,j,\nu} - \xi_{+\mu}\xi_{-\nu} - \xi_{-\mu}\xi_{+\nu} \tag{2.22}$$

For the transverse polarizations ward identity is ok. $\xi_\pm$ are problematic.

Exercise Calculate $i(M_{1+2} + M_3)^{\mu\nu}k_{2\nu}\xi_{-\mu}$

At tree level we could avoid the problem by just summing over the right polarizations, but the problem would come back at loop level, and would ultimately break the optical theorem, as show in Fig. 3.

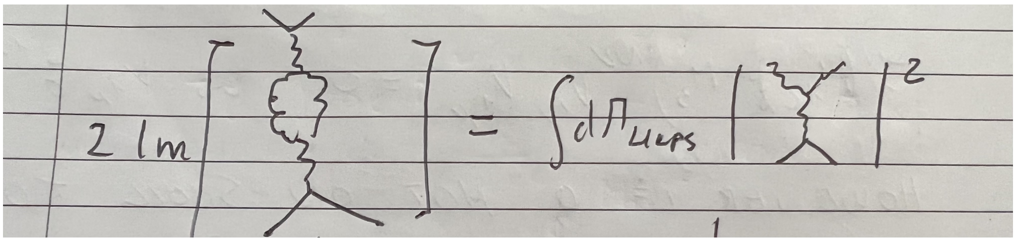

Figure 3: Optical theorem for $qq \to gg$

2.3 Faddev-Popov Lagrangian and BRST symmetry

This subsection is here only for completeness, and will not be discussed at lecture

We need to find a way to quantise the theory that cancels they unphysical d.o.f. from the amplitudes, so that theory can be renormalisable.

It turns out we can solve this problem by adding some additional fields in the adjont representation. However, for this to wok out we need the fields to be anticommuting fields that are Grassman variables (anticommuting with c-numbers). The requirement for this can be easily seen with functional integration, where adding a gauge-fixing term gives rise to additional terms as

$$\det \frac{\delta G(A^a)}{\delta \alpha} \tag{2.23}$$

is not independent of A for a non-abelian gauge group. One has

$$\frac{\delta G(A^a)}{\delta \alpha} = \frac{1}{g} \partial^\mu D_\mu \tag{2.24}$$

and one can write

$$\det \frac{\delta G(A^a)}{\delta \alpha} = \det \frac{1}{g} \partial^\mu D_\mu = \int \mathcal{D}c \mathcal{D}\bar{c} e^{i \int d^4x \bar{c}^a (-\partial^\mu D_\mu^{ac} c^c)} \tag{2.25}$$

$$D_\mu^{ac} = \partial_\mu \delta^{ac} + ig(t^b)_{ac} A_\mu^b \tag{2.26}$$

$$= \partial_\mu \delta^{ac} + g f_{abc} A_\mu^b \tag{2.27}$$

BRST symmetry The gauge-fixing term breaks gauge invariance. We want to add a gauge fixing term and retaining some kind of symmetry in the lagrangian, that will allow the theory to be renormalisable. This is the BRST symmetry. On the physical fields, it acts exactly the same as a gauge transformation with $\alpha^a = \epsilon c^a$:

$$\delta A_\mu^a = \epsilon D_\mu^{ac} c^c \tag{2.28}$$

$$\delta \Psi = i g \epsilon c^a t^a \Psi \tag{2.29}$$

We add 3 new fields. The 2 anticommuting fields $c, \bar{c}$, and an auxiliary field B. Auxiliary field means that it will have no kinetic term, so it will not propagate and it is unphysical. We will remove it later on from the lagrangian using equations of motion.

$$\delta c^a = -\frac{1}{2} g \epsilon f_{abc} c^b c^c \tag{2.30}$$

$$\delta \bar{c}^a = \epsilon B^a \tag{2.31}$$
$$\delta B^a = 0 \tag{2.32}$$

The infinitesimal variations for the new fields are defined using the following logic. We want

$$\delta \mathcal{L} = 0. \tag{2.33}$$

For this to happen, we need δ^2 to be zero for any field. So the definition of δc^a comes from the requirement $\delta^2 \Psi = 0$, while the definition of δB^a comes from te requirement $\delta^2 \bar{c}^a = 0$. The definition of $\delta \bar{c}$ is free and we just made a generic choice to set it equal to the auxiliary field B^a that we will later remove from the lagrangian.

Exercise Prove that $\delta^2 = 0$ for all fields.

We want a lagrangian that is invariant under BRST symmetry. The Yang-Mills part is already invariant due to gauge symmetry

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{GF} \tag{2.34}$$
$$\delta \mathcal{L}_{YM} = 0 \tag{2.35}$$

So we need to choose $\mathcal{L}_{GF}$ such that

$$\delta \mathcal{L}_{GF} = 0 \tag{2.36}$$

Thanks to the property

$$\delta^2 = 0 \tag{2.37}$$

this property will be satisfied by any term such that

$$\mathcal{L}_{GF} = \frac{1}{\epsilon} \delta O \tag{2.38}$$

for a generic operator O. We choose

$$O = \bar{c}^a \left(\frac{1}{2} \xi B^a - G^a \right) \tag{2.39}$$
$$G^a = \partial^\mu A^a_\mu \tag{2.40}$$

Exercise Show that

$$\mathcal{L}_{GF} = \frac{\xi}{2} (B^a)^2 - B^a \partial^\mu A^a_\mu - \bar{c}^a \partial^\mu D^{ac}_\mu c^c \tag{2.41}$$

Exercise Integrate out B^a to get

$$\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 - \bar{c}^a \partial^\mu D^{ac}_\mu c^c \tag{2.42}$$

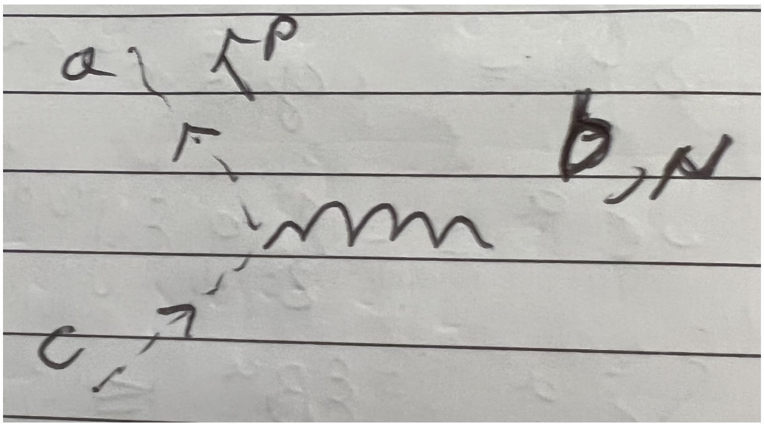

Figure 4: Feynman rule vertex for ghosts

The first term is the same as in QED, while the second term contains the ghost fields that interact with gluons. Note that the interaction is proportional to the structure constants f_{abc}, this is why we do not need ghosts in QED: they decouple form the physical particles.

The Feynman rule associated with the vertex in Fig. 4 is

$$-g f_{abc} p^{\mu}, \tag{2.43}$$

and the propagator is

$$\frac{i\delta^{ac}}{k^2}. \tag{2.44}$$

One also needs to add a $-$ sign for each ghost loop.

2.4 RGE equation for QED

This subsection is here only for completeness, and will not be discussed at lecture, apart from the last part with the final result

When making renormalization at 1 loop, we noticed that log functions appear in the observables (i.e., they are physical). For example, we saw that the propagator can be taken as the sum

$$D(q^2) = \frac{-i(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})}{q^2} \sum_{n=0}^{\infty} \frac{q^2 \Pi(q^2)}{q^2} = \frac{-i(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2})}{q^2(1-\Sigma(q^2))} \tag{2.45}$$

$$\Pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 dx x(1-x) \log\left(\frac{m^2 - q^2 x(1-x)}{m^2 - q_0^2 x(1-x)}\right) \tag{2.46}$$

where we have chosen a scale q_0 where

$$\Pi(q_0^2) = 0 \tag{2.47}$$

Now, in the limit where our scales are much larger than the mass of the fermion in the loop, this simplifies to

$$\Pi(q^2) = \frac{e^2}{2\pi^2}\log\left(\frac{q^2}{q_0^2}\right)\int_0^1 dx x(1-x) = \frac{e^2}{12\pi^2}\log\left(\frac{q^2}{q_0^2}\right) \tag{2.48}$$

we have anticipated that, at 1 loop, our EM potential gets modified to

$$V(p^2) = \frac{e^2}{p^2}(1+\Sigma(q^2)) \tag{2.49}$$

but if we resum all 1-loop 1PI diagrams we get instead

$$V(p^2) = \frac{e^2}{p^2}\frac{1}{1-\Pi(q^2)} \tag{2.50}$$

This tells us that it is important to resum all the 1PI diagrams, as when $\Sigma \to 1$, the result can be very different! Perturbation theory breaks down due to the large logs.

Now we will see a technique that will let us resum the large logs, without having to compute all the loop diagrams. We want to leave the computation of loop diagrams to the case where we want to make predictions to the NLO in α, but accounting for all $(\alpha \log p^2)^n$ terms. Rescaling the fields in our lagrangian we get

$$-\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - eZ_1\bar{\psi}A\!\!\!/\psi \tag{2.51}$$

Let's consider $G^{(2,1)}$, the 3 point function with 2 external fermions and one external photon. If we vary a bit the renormalization scale M, we get that

$$\psi \to \sqrt{Z_2}\psi \tag{2.52}$$

$$A \to \sqrt{Z_3}A \tag{2.53}$$

From the definition of the 3 point function

$$\langle\Omega|\bar{\psi}A_\mu\psi|\Omega\rangle \tag{2.54}$$

then

$$dG^{(2,1)} = (2\frac{\partial\sqrt{Z_2}}{\partial M} + \frac{\partial\sqrt{Z_3}}{\partial M})G^{(2,1)}\delta M \tag{2.55}$$

However, if we read $G^{(2,1)}$ from the lagrangian, we can think it as a function of M, e, and we can write

$$dG^{(2,1)} = (\frac{\partial G^{(2,1)}}{\partial M} + \frac{\partial G^{(2,1)}}{\partial e}\frac{\partial e}{\partial M})\delta M \tag{2.56}$$

Equating, and setting $G^{(2,1)} = eZ_1$ we get

$$\frac{e\partial Z_1}{\partial M} + Z_1\frac{\partial e}{\partial M} - (2\frac{\partial\sqrt{Z_2}}{\partial M} + \frac{\partial\sqrt{Z_3}}{\partial M})eZ_1 = 0 \tag{2.57}$$

Taking the first nonzero order in e, δ_i, we get

$$e\frac{\partial\delta_1}{\partial M} + \frac{\partial e}{\partial M} - e\frac{\partial\delta_2}{\partial M} - e\frac{1}{2}\frac{\partial\delta_3}{\partial M} = 0 \tag{2.58}$$

We get

$$\beta_e(e) = M\frac{\partial e}{\partial M} = eM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right) \tag{2.59}$$

We will now find the counterterms to calculate $\beta_e(e)$.

2.4.1 Vacuum polarization

As we said, we do not need to include the fermion masses. So the diagram is

$$
\begin{aligned}
i\Pi^{\mu\nu}(q^2) &= -(-ie)^2 \int \frac{d^dk}{(2\pi)^d} \frac{Tr[\gamma^\mu \not{k} \gamma^\nu (\not{k}+\not{q})]}{k^2(k+q)^2} && (2.60)\\
&= -4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{2k^\mu k^\nu - g^{\mu\nu}(k^2+kq)}{((k+xq)^2-\Delta)^2} && (2.61)
\end{aligned}
$$

where we dropped all q^μ, q^ν terms. We shift it

$$
\begin{aligned}
&= -4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{2l^\mu l^\nu - g^{\mu\nu}((l-xq)^2 - xq^2)}{(l^2-\Delta)^2} && (2.62)\\
&= -4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{2l^\mu l^\nu - g^{\mu\nu}(l^2 - x(1-x)q^2)}{(l^2-\Delta)^2} && (2.63)\\
&= -4e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\frac{2}{d}l^2 g^{\mu\nu} - g^{\mu\nu}(l^2+\Delta)}{(l^2-\Delta)^2} && (2.64)\\
&= 4e^2 g^{\mu\nu} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{(1-\frac{2}{d})l^2+\Delta}{(l^2-\Delta)^2} && (2.65)\\
&= \frac{4e^2 g^{\mu\nu}}{16\pi^2} \int_0^1 dx \left((1-\frac{2}{d})(-i)\frac{d}{2}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Delta^{1-\frac{d}{2}}} + i\frac{\Gamma\left(2-\frac{d}{2}\right)\Delta}{\Delta^{2-\frac{d}{2}}} \right) && (2.66)\\
&= \frac{4ie^2 g^{\mu\nu}}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx \Delta \frac{\left(-\frac{1-\frac{2}{d}}{1-\frac{d}{2}}\frac{d}{2}+1\right)}{\Delta^{2-\frac{d}{2}}} && (2.67)\\
&= \frac{8ie^2 g^{\mu\nu}}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx \Delta \frac{1}{\Delta^{2-\frac{d}{2}}} && (2.68)\\
&= -q^2\frac{8ie^2 g^{\mu\nu}}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx x(1-x) \frac{1}{(-q^2x(1-x))^{2-\frac{d}{2}}} && (2.69)
\end{aligned}
$$

The counterterm will need to cancel the divergency, so will need to be

$$
\begin{aligned}
(-iq^2 g^{\mu\nu})\delta_3 &= +q^2\frac{8ie^2 g^{\mu\nu}}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right) \int_0^1 dx x(1-x) \frac{1}{(M^2x(1-x))^{2-\frac{d}{2}}} && (2.70)\\
&= q^2\frac{8ie^2 g^{\mu\nu}}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right) \frac{1}{(M^2)^{2-\frac{d}{2}}} \int_0^1 dx x(1-x) && (2.71)\\
&= (-iq^2 g^{\mu\nu})(-)\frac{e^2}{12\pi^2}\Gamma\left(2-\frac{d}{2}\right) \frac{1}{(M^2)^{2-\frac{d}{2}}} && (2.72)\\
\delta_3 &= -\frac{e^2}{12\pi^2}\Gamma\left(2-\frac{d}{2}\right) \frac{1}{(M^2)^{2-\frac{d}{2}}} && (2.73)
\end{aligned}
$$

2.4.2 Fermion Self Energy

From the fermion self energy we get

$$
i\Sigma(q^2) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{i\not{k}}{k^2} \gamma^\nu \frac{-ig_{\mu\nu}}{(k-q)^2} \qquad (2.74)
$$

$$= 2e^2 \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}}{k^2(k-q)^2} \tag{2.75}$$

$$= 2e^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}}{((k-xq)^2-\Delta)^2} \tag{2.76}$$

$$= 2e^2 \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{x\not{q}}{(l^2-\Delta)^2} \tag{2.77}$$

$$= 2e^2 \not{q} \int_0^1 dx x \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2-\Delta)^2} \tag{2.78}$$

$$= \frac{2e^2 i}{16\pi^2} \not{q} \int_0^1 dx x \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \tag{2.79}$$

So the counterterm is

$$i\not{q}\delta_2 = -\frac{2e^2 i}{16\pi^2} \not{q} \int_0^1 dx x \frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}} \tag{2.80}$$

$$\delta_2 = -\frac{2e^2}{16\pi^2} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}} \int_0^1 dx x \tag{2.81}$$

$$= -\frac{e^2}{16\pi^2} \frac{\Gamma\left(2-\frac{d}{2}\right)}{(M^2)^{2-\frac{d}{2}}} \tag{2.82}$$

2.4.3 Vertex Function

$$i\Gamma^\mu = (-ie)^3 \int \frac{d^dk}{(2\pi)^d} \gamma^\nu \frac{i\not{k}}{k^2} \gamma^\mu \frac{i(\not{k}+\not{q})}{(k+q)^2} \gamma^\rho \frac{-ig_{\nu\rho}}{k^2} \tag{2.83}$$

$$= 2e^3 \int \frac{d^dk}{(2\pi)^d} \frac{\not{k}\gamma^\mu(\not{k}+\not{q})}{k^4(k+q)^2} \tag{2.84}$$

We need to decompose the product of gamma matrices and find the coefficient of γ^μ, so we project on it

$$\frac{1}{16} Tr[\not{k}\gamma^\mu(\not{k}+\not{q})\gamma_\mu] = -\frac{2}{16} Tr[\not{k}(\not{k}+\not{q})] = -\frac{1}{2}(k^2+kq) \tag{2.85}$$

So

$$i\Gamma^\mu = -e^3\gamma^\mu \int \frac{d^dk}{(2\pi)^d} \frac{k^2+kq}{k^4(k+q)^2} \tag{2.86}$$

$$= -2e^3\gamma^\mu \int_0^1 dx \int_0^{1-x} dy \int \frac{d^dk}{(2\pi)^d} \frac{k^2+kq}{((k+xq)^2-\Delta)^3} \tag{2.87}$$

$$= -2e^3\gamma^\mu \int_0^1 dx(1-x) \int \frac{d^dl}{(2\pi)^d} \frac{(l-xq)^2-xq^2}{(l^2-\Delta)^3} \tag{2.88}$$

$$= -2e^3\gamma^\mu \int_0^1 dx(1-x) \int \frac{d^dl}{(2\pi)^d} \frac{l^2-x(1-x)q^2}{(l^2-\Delta)^3} \tag{2.89}$$

$$= -2e^3\gamma^\mu \int_0^1 dx(1-x) \int \frac{d^dl}{(2\pi)^d} \frac{l^2+\Delta}{(l^2-\Delta)^3} \tag{2.90}$$

The second term is finite and does not give raise to $\Gamma(2-d/2)$, so we drop it.

$$i\Gamma^\mu \sim -2e^3\gamma^\mu \int_0^1 dx(1-x) \int \frac{d^dl}{(2\pi)^d} \frac{l^2}{(l^2-\Delta)^3} \tag{2.91}$$

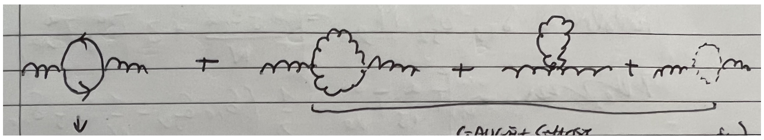

Figure 5: Feynman Diagrams for Gluon Self Energy

$$= -\frac{2ie^3}{16\pi^2}\gamma^\mu \Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx(1-x)\frac{1}{\Delta^{2-\frac{d}{2}}} \tag{2.92}$$

So the counterterm is

$$\begin{aligned}
-ie\gamma^\mu\delta_1 &= \frac{2ie^3}{16\pi^2}\gamma^\mu \Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx(1-x)\frac{1}{(M^2)^{2-\frac{d}{2}}} && (2.93)\\
\delta_1 &= -\frac{2e^2}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx(1-x)\frac{1}{(M^2)^{2-\frac{d}{2}}} && (2.94)\\
&= -\frac{2e^2}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right)\frac{1}{(M^2)^{2-\frac{d}{2}}}\int_0^1 dx(1-x) && (2.95)\\
&= -\frac{e^2}{16\pi^2}\Gamma\left(2-\frac{d}{2}\right)\frac{1}{(M^2)^{2-\frac{d}{2}}} && (2.96)
\end{aligned}$$

2.4.4 Running of electron coupling constant

We obtain

$$\begin{aligned}
\beta_e(e) &= M\frac{\partial e}{\partial M} = eM\frac{\partial}{\partial M}\left(-\delta_1+\delta_2+\frac{1}{2}\delta_3\right) = \frac{e^3}{12\pi^2} && (2.97)\\
\beta_\alpha &= M\frac{\partial \alpha}{\partial M} = \frac{e}{2\pi}\beta_e = \frac{e^4}{24\pi^3} = \frac{2}{3\pi}\alpha^2 && (2.98)\\
\frac{d\alpha}{d\log M^2} &= \frac{1}{2}\beta_\alpha = \frac{1}{3\pi}\alpha^2 && (2.99)
\end{aligned}$$

We can now solve this differential equation

$$\begin{aligned}
\frac{d\alpha}{\alpha^2} &= \frac{1}{3\pi}d\log M^2 && (2.100)\\
[\frac{1}{\alpha}]^{\alpha(q^2)}_{\alpha(M_0^2)} &= \frac{1}{3\pi}\log\frac{q^2}{M_0^2} && (2.101)\\
\alpha(q^2) &= \frac{\alpha(M_0^2)}{1-\frac{\alpha(M_0^2)}{3\pi}\log\frac{q^2}{M_0^2}} && (2.102)
\end{aligned}$$

2.5 Asymptotic freedom, Running α

This subsection is here only for completeness, and will not be discussed at lecture, apart from the last part with the final result

We now want to calculate the RGE equation for a non-abelian gauge group. The first step is to calculate the gluon Self energy. This includes non only the QED-like diagram with a fermion loop, but also new diagrams that arise in non-abelian gauge groups, as shown in Fig. 5.

The QED-like diagram can be quickly worked up by analogy with QED, the only difference is the presence of the group generator t^a at each vertex, so the contribution to δ_3 will be

$$\delta_{3,QED} = \frac{-8e^2}{6(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} \tag{2.103}$$

$$\delta_{3,ferm,QCD} = \delta_{3,QED}|_{e\to g} \times Tr[t^a t^b] \tag{2.104}$$

2.5.1 Color Traces

We will need to be able to calculate color traces to be able to compute any diagram in QCD, in particular the ones we need for the running of α. The single generator is traceless

$$Tr[t^a] = 0 \tag{2.105}$$

Each generator is "orthogonal" to every other

$$Tr[t^a_r t^b_r] = C(r)\delta^{ab} \tag{2.106}$$

where $C(r)$ is a number that depends on the representation r. Quarks are in the fundamental representation, so are our t^a, and

$$C(F) = \frac{1}{2} \tag{2.107}$$

One important operator to consider is the Quadratic Casimir operator, sometimes just shortened to Casimir operator,

$$t^a_r t^a_r \tag{2.108}$$

One can check that this operator commutes with all generators of the group.

Exercise Prove this

Due to Schur's lemma, it means that it is proportional to the identity

$$t^a_r t^a_r = C_2(r)\mathbb{1}_{d(r)\times d(r)} \tag{2.109}$$

By taking the trace on both sides one gets

$$Tr[t^a_r t^a_r] = C(r)\delta^{aa} = C_2(r)d(r) \tag{2.110}$$

$$C(r)d(G) = C_2(r)d(r) \tag{2.111}$$

For the fundamental of $SU(N)$ one has

$$C(F) = \frac{1}{2} \tag{2.112}$$

$$d(G) = N^2 - 1 \tag{2.113}$$

$$d(F) = N \tag{2.114}$$

$$C_2(F) = \frac{N^2-1}{2N} \tag{2.115}$$

giving $C_2(F) = 3/4$ for $SU(2)$ and $C_2(F) = 4/3$ for $SU(3)$.

An useful identity to calculate color traces is

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta jk - \frac{1}{N}\delta_{ij}\delta kl\right) \tag{2.116}$$

Proof: Let M be an $N \times N$ hermitian matrix, then

$$M = M_0\mathbb{1} + M_a t^a \tag{2.117}$$

$$M_0 = \frac{1}{N}Tr[M] \tag{2.118}$$

$$M_a = 2Tr[Mt^a] \tag{2.119}$$

$$M = \frac{1}{N}Tr[M]\mathbb{1} + 2Tr[Mt^a]t^a \tag{2.120}$$

$$M_{ij} = \frac{M_{kk}\delta_ij}{N} + 2M_{kl}t^a_{kl}t^a_{ij} \tag{2.121}$$

$$\delta_{il}\delta_{jk}M_{lk} = \left(\frac{\delta_{ij}\delta_{kl}}{N} + 2t^a_{ij}t^a_{kl}\right)M_{lk} \tag{2.122}$$

obtaining

$$\left(t^a_{ij}t^a_{kl} - \frac{1}{2}\left(\delta_{il}\delta jk - \frac{1}{N}\delta_{ij}\delta kl\right)\right)M_{kl} = 0 \tag{2.123}$$

$$\left(t^a_{ij}t^a_{kl} - \frac{1}{2}\left(\delta_{il}\delta jk - \frac{1}{N}\delta_{ij}\delta kl\right)\right) = 0 \tag{2.124}$$

Exercise Use the identity to calculate

$$Tr[t^a t^b t^a t^c] \tag{2.125}$$

Exercise One can write the identity

$$T^aT^b = \frac{1}{2}\left(\frac{1}{N}\delta_{ab}\mathbb{1} + (d_{abc} + if_{abc})T^c\right) \tag{2.126}$$

find the value of d_{abc}.

Exercise Calculate $f_{abc}f_{abd}$

Note $f_{abc}f_{abd}$ is the quadratic casimir operator for the adjoint representation,

$$C_2(G) = C(G) = N \tag{2.127}$$

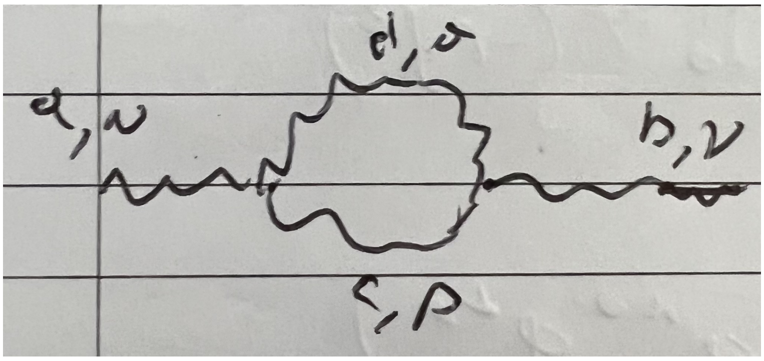

Figure 6: Feynman Pure Gauge Diagram for Gluon Self Energy 1

2.5.2 Gluon self energy - δ_3

Coming back to the gluon self energy, the contribution of fermion loops to δ_3 is

$$\delta_{3,ferm} = \frac{-8g^2}{6(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} C(r)\delta^{ab} \tag{2.128}$$

$$= \left(\frac{g^2}{(4\pi)^2} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-\frac{d}{2}}} \right) \left(-\frac{4}{3} C(r) \right) \tag{2.129}$$

We now need to calculate the pure gauge and ghost contributions.

We start from the diagram in Fig.6. We choose the gauge $\xi = 1$.

$$= \frac{1}{2} \int \frac{d^4}{(2\pi)^4} \frac{-i}{p^2} \frac{-i}{(p+q)^2} g^2 f_{acd} f_{bcd} \tag{2.130}$$

$$= -\frac{g^2}{2} C_2(G)\delta^{ab} \int_0^1 dx \int \frac{d^4P}{(2\pi)^4} \frac{N^{\mu\nu}}{(P^2-\Delta)^2} \tag{2.131}$$

We can discard any term odd in P inside $N^{\mu\nu}$, and substitute $P^\mu P^\nu \to \frac{P^2}{d} g^{\mu\nu}$, obtaining

$$N^{\mu\nu} = -g^{\mu\nu} p^2 (1 - \frac{1}{d}) - g^{\mu\nu} q^2 ((2-x)^2 + (1+x)^2) \tag{2.132}$$

$$+ \; q^\mu q^\nu ((2-d)(1-2x)^2 + 2(1+x)(2-x)) \tag{2.133}$$

We can now perform the wick rotation and make the 4-d integral

$$= \frac{ig^2}{(4\pi)^{d/2}} C_2(G)\delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}} \Bigg(\Gamma(1-\frac{d}{2}) g^{\mu\nu} q^2 (\frac{3}{2}(d-1)x(1-x)) \tag{2.134}$$

$$+ \; \Gamma(2-\frac{d}{2}) g^{\mu\nu} q^2 (\frac{1}{2}(2-x)^2 + \frac{1}{2}(1+x)^2 \tag{2.135}$$

$$- \; \Gamma(2-\frac{d}{2}) q^\mu q^\nu ((1-\frac{d}{2})(1-2x)^2 + (1+x)(2-x))) \Bigg) \tag{2.136}$$

This diagram alone does not have the structure that we desire. It is not transverse, and we have a pole at $d = 2$ (mass renormalization).

We evaluate the second diagram, as shown in Fig. 7,

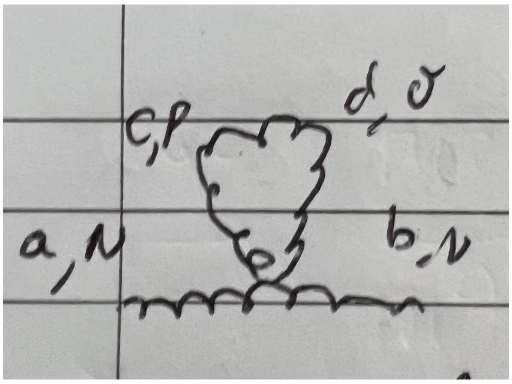

Figure 7: Feynman Pure Gauge Diagram for Gluon Self Energy 2

$$= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{-ig^{\mu\nu}}{p^2}\delta^{cd}(-ig^2)\left(f_{abe}f_{cde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \right. \tag{2.137}$$

$$+ \; f_{ace}f_{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f_{ade}f_{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})) \tag{2.138}$$

First line is killed by asymmetry of f, second and third line are equal, obtaining

$$= (-g^2)C_2(G)\delta^{ab}g^{\mu\nu}(d-1)\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2} \tag{2.139}$$

We want to sum it with the other diagram, so we perform the same shift,

$$= -g^2 C_2(G)\delta^{ab}g^{\mu\nu}(d-1)\int_0^1 dx \int \frac{d^4P}{(2\pi)^4}\frac{P^2 + (1-x)^2q^2}{(P^2-\Delta)^2} \tag{2.140}$$

$$= \frac{ig^2}{(4\pi)^{d/2}}\delta^{ab}C_2(G)\int_0^1 dx \frac{1}{\Delta^{2-d/2}}\left(-\Gamma(1-\frac{d}{2})g^{\mu\nu}q^2(\frac{1}{2}d(d-1)x(1-x)) \right. \tag{2.141}$$

$$\left. - \; \Gamma(2-\frac{d}{2})g^{\mu\nu}q^2(d-1)(1-x)^2\right) \tag{2.142}$$

We could now wonder if the pole in $d = 2$ gets canceled once summing up the 2 diagrams. We can try to verify this

$$= \Gamma(1-\frac{d}{2})g^{\mu\nu}q^2\left(\frac{3}{2}(d-1)x(1-x) - \frac{1}{2}d(d-1)x(1-x)\right) \tag{2.143}$$

$$= \Gamma(1-\frac{d}{2})g^{\mu\nu}q^2\frac{(d-1)x(1-x)(3-d)}{2} \tag{2.144}$$

Indeed, the pole does not cancel.

This is how we realise we really need the additional loop with ghosts,

$$= (-1)\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2}\frac{i}{(p+q)^2}g^2 f_{dac}f_{cbd}(p+q)^\mu p^\nu \tag{2.145}$$

$$= \frac{ig^2}{(4\pi)^{d/2}}\delta^{ab}C_2(G)\int_0^1 dx\frac{1}{\Delta^{2-d/2}}\left(-\Gamma(1-\frac{d}{2})g^{\mu\nu}q^2\frac{x(1-x)}{2} + \Gamma(2-\frac{d}{2})q^\mu q^\nu x(1-x)\right) \tag{2.146}$$

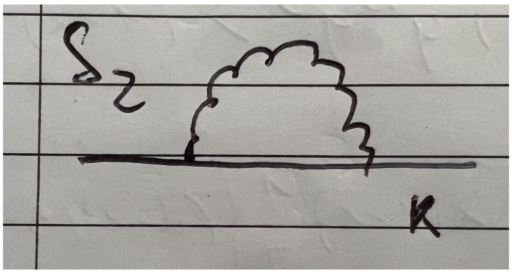

Figure 8: Fermion Self Energy

Let's now check again the cancellation of the pole at $d = 2$, now we get

$$= \Gamma(1 - \frac{d}{2}) g^{\mu\nu} q^2 x(1-x) \frac{(d-1)(3-d)-1}{2} \tag{2.147}$$

$$= \Gamma(1 - \frac{d}{2}) g^{\mu\nu} q^2 x(1-x)(1 - \frac{d}{2})(d-2) \tag{2.148}$$

$$= \Gamma(2 - \frac{d}{2}) g^{\mu\nu} q^2 x(1-x)(d-2) \tag{2.149}$$

Good news, the pole cancels! We now have to work out the coefficient of $\Gamma(2 - \frac{d}{2})$. If we sum up and integrate over x we get that the result is transverse!

$$= \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \left(i(q^2 g^{\mu\nu} - q^\mu q^\nu)\right) \delta^{ab} \left(C_2(G) \frac{14-d}{6}\right) \tag{2.150}$$

We are finally ready to compute δ_3:

$$\delta_3 = \frac{g^2}{(4\pi)^2} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \left(\frac{5}{3} C_2(G) - \frac{4}{3} C(r) n_f\right) \tag{2.151}$$

Exercise Perform the same calculation in a generic ξ gauge.

2.5.3 Quark self energy - δ_2

We can work the diagram in Fig. 8 out in analogy with QED

$$\delta_{2,QED} = -\frac{e^2}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \frac{1}{(M^2)^{2-d/2}} \tag{2.152}$$

so we can get the QCD one as

$$\delta_{2,QCD} = \delta_{2,QED}|_{e\to g} t^a t^b \delta^{ab} = \delta_{2,QED}|_{e\to g} C_2(r) \tag{2.153}$$

So we get

$$\delta_2 = \frac{g^2}{(4\pi)^2} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \left(-C_2(r)\right) \tag{2.154}$$

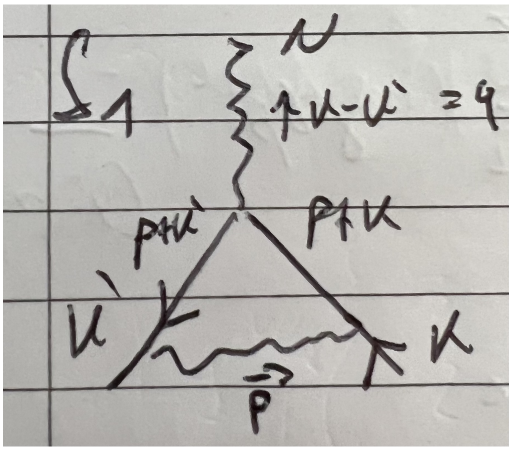

Figure 9: Vertex Correction 1

2.5.4 QCD vertex correction - δ_1

We have 2 diagrams in this case.

The first one is similar to QED, and we will work in analogy. It is shown in Fig. 9. We only need the divergent part of this diagram. The result from QED was that

$$\delta_{1,QED} = -\frac{e^2}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \tag{2.155}$$

Working by analogy we have

$$\delta_{1,ferm,QCD} = \delta_{1,QED}|_{e\to g} t^b t^a t^b \tag{2.156}$$

We work out the color factor as

$$= t^b t^a t^b = t^b t^b t^a + t^b[t^a, t^b] \tag{2.157}$$

$$= C_2(r)t^a + it^b f_{abc}t^c = C_2(r)t^a + \frac{i}{2} f_{abc}[t^b, t^c] \tag{2.158}$$

$$= C_2(r)t^a + \frac{1}{2} i f_{abc} i f_{bcd} t^d = \left(C_2(r) - \frac{1}{2}C_2(G)\right) t^a \tag{2.159}$$

So the contribution of the diagram reads (the t^a factor is reabsorbed when factoring out the feynman rule for the counterterm)

$$\frac{g^2}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}}\left(-C_2(r) + \frac{1}{2}C_2(G)\right) \tag{2.160}$$

The second diagram, shown in Fig. 10, is,

$$= \int \frac{d^4 p}{(2\pi)^4}(ig\gamma_\nu t^b)\frac{i}{p^2}(ig\gamma_\rho t^c)\frac{-i}{(k'-p)^2}\frac{-i}{(k-p)^2} g f_{abc} \tag{2.161}$$

$$\times \left(g^{\mu\nu}(2k' - k - p)^\rho + g^{\nu\rho}(-k' - k + 2p)^\mu + g^{\mu\rho}(2k - k' - p)^\nu\right) \tag{2.162}$$

$$\to \frac{g^3}{2}C_2(G)t^a \int \frac{d^4 P}{(2\pi)^4}\gamma_\nu \not{P} \gamma_\rho \frac{g^{\mu\nu}P^\rho - 2g^{\nu\rho}P^\mu + g^{\mu\rho}P^\nu}{(P^2-\Delta)^3} \tag{2.163}$$

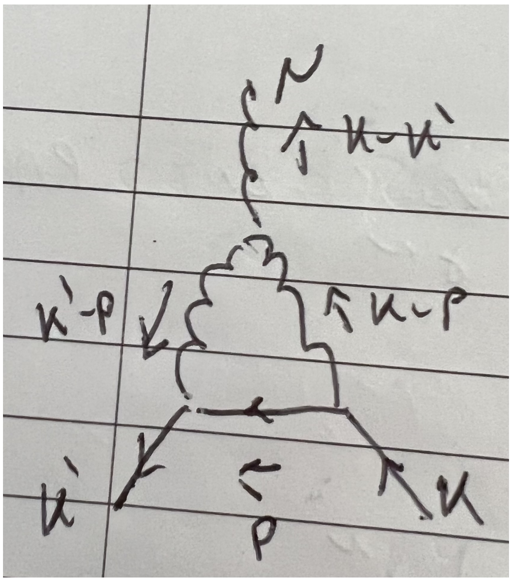

Figure 10: Vertex Correction 2

$$= \frac{g^3}{2} C_2(G) t^a \gamma^\mu \int \frac{d^4P}{(2\pi)^4} \frac{(2+\frac{4}{d})P^2}{(P^2-\Delta)^3} \tag{2.164}$$

$$= (-ig\gamma^m u t^a) \frac{g^2}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \left(-\frac{3}{2} C_2(G)\right) \tag{2.165}$$

So we can now get δ_1:

$$\delta_1 = \frac{g^2}{(4\pi)^2} \frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \left(-C_2(r) - C_2(G)\right) \tag{2.166}$$

2.5.5 RGE for α

We are now ready to compute the QCD beta function.

$$\beta(g) = gM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\right) \tag{2.167}$$

$$= \frac{g^3}{(4\pi)^2}\Gamma(2-\frac{d}{2})(-2)(2-\frac{d}{2})\left(C_2(r) + C_2(G) - C_2(r) + \frac{1}{2}(\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r))\right) \tag{2.168}$$

$$= -\frac{g^3}{(4\pi)^2}\Gamma(3-\frac{d}{2})\left(\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right) \tag{2.169}$$

$$= -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right) \tag{2.170}$$

Fr an abelian group, $C_2(G) = 0$ and $\beta > 0$. FOr a non abelian group, $\beta(g) < 0$ for small n_f. For the QCD case

$$C_2(G) = N = 3 \tag{2.171}$$

$$C(F) = \frac{1}{2} \tag{2.172}$$

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(11 - \frac{2}{3}n_f\right) \tag{2.173}$$

Exercise Evolve α_s from M_Z to find the QCD pole.

2.5.6 Renormalizability and gauge invariance

Now one could ask himself: should I run the same calculation for the 3-gluon and 4-gluon point functions, and check that I get the same RGE equation for g_s also for them? In a theory that is gauge invariant, this is indeed not necessary. As gauge invariance is not spoiled at loop level, it will still hold after renormalization, and so will the relations between the coefficients of the various vertices.

It is a general result, that gauge invariance plus having in the lagrangian all operators with $D \leq 4$ compatible with the chosen set of symmetries, allows the theory to be renormalizable at all orders. This, in turn, allows the theory to be predictive for precision observables, thanks to the fact that at any order in perturbation theory, one will need only the counterterms coming from the tree level lagrangian. his means that one only needs to perform a number of measuremenets equal to the number of paramaters of the bare lagrangian, and loop corrections will deliver precision predictions about the theory.

In a non-renormalisable theory, at every order in perturbation theory, new operators get added to the lagrangian, requiring to add more counterterms and perform more measurements at each stage. The theory can still make predictions at low energies, however one can always fix any disagreement of the theory with experiment by going one level more in perturbation theory and fitting any new experimental data that was not agreeing with the model at N loops with a_N parameters, with the model at $N+1$ loops and $a_{N+1} > a_N$ paremeters.

2.6 Number of Colors

How do we know that the number of colors is 3? There are several experimental probes about the number of colors. These include

1. The spin-statistics wave function of the Δ^{++} baryon, that needs to be completely anti-symmetric, but is instead symmetric in L, S and flavour. Adding the the color degree of freedom, only for $N_c = 3$ the function can be a completely anti-symmetric singlet.

2. The ratio of $\sigma(e^+e^- \to hadrons)$ to $\sigma(e^+e^- \to \mu^+\mu^-)$, that is proportional to the number of colors N_c, as quarks appear in the final states

3. The value of the $\pi^0 \to \gamma\gamma$ decay rate, that happens through a quark loop, and as quarks appear only as internal states, and therefore the process is proportional to N_c^2

3 Young Mills Lagrangian, Mesons, Baryons

3.1 Quantum Symmetries of YM lagrangian

We want to study symmetries of YM theories, for example QCD

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} + (\bar{u}, \bar{d}, \bar{s})i\not{D}\begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{3.1}$$

We can rewrite the lagrangian decomposing in left and right handed fields

$$q_i = \left(\left(\frac{1+\gamma^5}{2}\right) + \left(\frac{1-\gamma^5}{2}\right)\right) q_i = \left(\frac{1+\gamma^5}{2}\right) q_i + \left(\frac{1+\gamma^5}{2}\right) q_i = q_{i,R} + q_{i,L} \tag{3.2}$$

We get the following lagrangian

$$(\bar{u}, \bar{d}, \bar{s})i\not{D}\begin{pmatrix} u \\ d \\ s \end{pmatrix} = (\bar{u}_R, \bar{d}_R, \bar{s}_R)i\not{D}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} + (\bar{u}_L, \bar{d}_L, \bar{s}_L)i\not{D}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \tag{3.3}$$

The left and right fields do not talk to each other. Such a theory is invariant under the global symmetry of the group

$$U(3)_L \otimes U(3)_R = U(1)_L \otimes U(1)_R \otimes SU(3)_L \times SU(3)_R \tag{3.4}$$

The $U(1)$ factors transform all fields of the given helicity and does not act on the fields of the other helicity. The $SU(3)$ factors apply $SU(3)$ transformations to the fields of the given helicity (thus mixing up different flavours) and do not act on the fields of the other helicity.

$$q_L \to e^{i\alpha} q_L \quad U(1)_L \tag{3.5}$$

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \to U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \quad SU(3)_L \tag{3.6}$$

The symmetry group can be rewritten as

$$U(3)_V \otimes U(3)_A = U(1)_V \otimes U(1)_A \otimes SU(3)_V \times SU(3)_A \tag{3.7}$$

$U(1)_V$ acts on all quarks with same phase shift

$$q \to e^{i\alpha} q \tag{3.8}$$

This symmetry is exact and is called $U(1)_B$ barion number conservation. The $SU(3)_V$ is an $SU(3)$ transformation acting in the same way for left and right handed components

$$Q \to UQ \tag{3.9}$$

$$Q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{3.10}$$

This symmetry would be exact even with quark masses turned on, as long as they are all degenerate. So it is broken by quark mass differences, $m_d - m_u$, $2m_s - m_d - m_u$. This (approximate) symmetry is seen in nature and is called $SU(3)_F$. The subgroup with only the u, d quarks is called $SU(2)_F$ or isospin symmetry. Before discussing the axial symmetries, we need to discuss the Goldstone theorem.

3.2 Goldstone Theorem and linear σ model

Let Φ be a real scalar field in the fundamental repr. of $SO(N)$. A Φ^4 theory invariant under $SO(N)$ can be written as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2}\mu^2\Phi^\dagger\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi)^2 \tag{3.11}$$

You may notice that the mass term has the wrong sign. No worries! We will see that the theory still makes perfectly sense. What will happen is a symmetry breaking, and some fields will need to be rewritten, as some of them will get a nonzero vev. Taking the potential, we find the minima condition

$$\frac{\partial V}{\partial \Phi} = -\mu^2\Phi^\dagger + \lambda(\Phi^\dagger\Phi)\Phi^\dagger = 0 \tag{3.12}$$

$$\rightarrow \Phi^\dagger\Phi = \frac{\mu^2}{\lambda} \tag{3.13}$$

Thus we need to have at least one component of the n-vector that has nonzero vev. Note that we can take to be just one component without loss of generality, because we can always rotate the vector by a $SO(N)$ transformation to have all but one components with no vev. This transformation will define the physical base. Thus

$$\langle\Phi\rangle = (0, ..., \sqrt{\frac{\mu^2}{\lambda}}) \tag{3.14}$$

From now on, we are not more allowed to perform a generic $SO(N)$ rotation, as the system is clearly not invariant anymore under such transformations. What remains is an $SO(N-1)$ symmetry for the first $N-1$ components of Φ. We will call them $\tilde{\Phi}$, while we will call ρ the last field component of Φ,

$$\Phi = (\tilde{\Phi}, \sqrt{\frac{\mu^2}{\lambda}} + \rho) \tag{3.15}$$

If we rewrite the lagrangian using such fields we get

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho \\ &+ \frac{1}{2}\mu^2\left(\tilde{\Phi}^\dagger\tilde{\Phi} + (\frac{\mu^2}{\lambda}\rho)^2\right) \\ &+ \frac{1}{4}\lambda\left(\tilde{\Phi}^\dagger\tilde{\Phi} + (\frac{\mu^2}{\lambda}\rho)^2\right)^2 \qquad (3.16)\\ &= \frac{1}{2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}(2\mu^2)\rho^2 + \mathcal{O}(S^3) \qquad (3.17)\end{aligned}$$

The field ρ now has a mass term with the right sign, while the remaining fields are massless and have e remnant $SU(N-1)$ symmetry!

Goldstone Theorem: If a Lagrangian, invariant under a set of continuus transformations, gets the symmetry spontaneously broken, there is a massless goldstone boson for each broken generator.

How many generators where broken?

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = N-1 \tag{3.18}$$

that is exactly the number of massless particles remaining in the theory!

PROOF We want to proove that, $\forall$ continuus symmetries of $\mathcal{L}$ that are not a symmetry of the minimum, then

$$M_{ab} = \frac{\partial V}{\partial\phi_a\partial\phi_b}(\phi_0) \tag{3.19}$$

has a zero eigenvalue. The symmetry can be expressed as a infinitesimal tranformation

$$\phi^a \to \phi^a + \alpha R^{ab}\phi^b, \quad \alpha \ll 1 \tag{3.20}$$

We take ϕ as constant, so a symmetry of $\mathcal{L}$ is a symmetry of V. The invariance of V means that

$$V(\phi^a) = V(\phi^a + \Delta\phi^a) \tag{3.21}$$

$$\frac{\partial V}{\partial\phi^a}(\phi)\Delta\phi^a(\phi) = \frac{\partial V}{\partial\phi^a}(\phi)\alpha R^{ac}\phi^c = 0 \tag{3.22}$$

We differentiate the previous relation w.r.t ϕ_b:

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi)R^{ac}\phi^c + \frac{\partial V}{\partial\phi^a}(\phi)R^{ab} = 0 \tag{3.23}$$

we now spcify to the point $\phi = \phi_0$,

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi_0)R^{ac}\phi_0^c + \frac{\partial V}{\partial\phi^a}(\phi_0)R^{ab} = 0 \tag{3.24}$$

$$\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}(\phi_0)R^{ac}\phi_0^c = M_{ab}R^{ac}\Phi_0^c = 0 \tag{3.25}$$

where the second equation is obtained using the fact that the gradient of the potential is zero at the minima. Now, the number of nonzero components $R^{ac}\phi_0^c$ is equal to the number of broken generators R^{ac}, i.e. generators for which

$$R^{ac}\Phi^c \neq 0 \tag{3.26}$$

These are all linearly independent components, and as all components of

$$M_{ab}R^{ac}\Phi_0^c \tag{3.27}$$

are required to be zero, it follows that the matrix M_{ab} needs to have the same number of zero eigenvalues as the number of broken generators.

Effective Potential: radiative (loop) corrections modify the shape of the potential. However, one can show that the Goldstone Theorem remains valid at all levels of perturbation theory.

3.3 Light quarks flavour symmetries and quasi-Goldstone bosons

We can now go back to consider the symmetries of our lagrangian. We need to acknowledge first an important fact: the fact that α_{QCD} hits a pole at low energy has the important consequence that quarks bilinears develop what is called quark condensate

$$\langle 0|\bar{q}_j q_i|0\rangle = \mathcal{O}(\Lambda^3_{QCD})\delta_{ij} \tag{3.28}$$

Such condensate can potentially break spontaneusly some of the symmetries of our lagrangian. Now we can start considering back the symmetries of our larangian. As we anticipated, $U(1)_V = U(1)_B$ is a symmetry of the lagrangian and remains unbroken. Regarding $SU(3)_V = SU(3)_F$, once again as we anticipated this symmetry is approximate. It is hardly broken by the mass differences of quarks, $|m_u - m_d|$ and $|m_s - \frac{m_u+m_d}{2}|$, but is not broken by the quark condensate, as

$$\mathcal{O}(\Lambda_{QCD}^3)\delta_{ij} = \langle 0|\bar{q}_j q_i|0\rangle \to \langle 0|\bar{q}_j U^\dagger U q_i|0\rangle = \langle 0|\bar{q}_j q_i|0\rangle = \mathcal{O}(\Lambda_{QCD}^3)\delta_{ij} \tag{3.29}$$

the symmetry is approximately realised as both quark mass differences are smaller than Λ_{QCD}, that is the relevant scale for the theory

$$|m_d - m_u| \ll \Lambda_{QCD}, \quad |m_s - \frac{m_u + m_d}{2}| \lesssim \Lambda_{QCD} \tag{3.30}$$

Once again, a the first mass difference is much smaller than Λ_{QCD}, the $SU(2)$ isospin symmetry is more exact than the flavour $SU(3)$. Now, we can start discussing the axial symmetries. $SU(3)_A$ is an approximate symmetry of the lagrangian, as it is hardly broken by the quark masses (not their differences), but, more importantly, it is spontaneously broken by the quark condensate, as

$$\mathcal{O}(\Lambda_{QCD}^3)\delta_{ij} = \langle 0|\bar{q}_j q_i|0\rangle \to \langle 0|\bar{q}_{R,j} U U q_{L,i}|0\rangle + \langle 0|\bar{q}_{L,j} U^\dagger U^\dagger q_{L,i}|0\rangle \neq \langle 0|\bar{q}_j q_i|0\rangle \tag{3.31}$$

As we have 8 broken generator, we would expect 8 goldstone bosons. But as the symmetry is only approximate, we expect 8 quasi-goldstone bosons, whose mass term is expected to be proportional to the quark masses, so that it would go to zero is the symmetry was a proper symmetry of the lagrangian, just spontaneously broken by the quark condensate. So we expect them to be "light". This is called the meson octet, shown in Fig. 11.

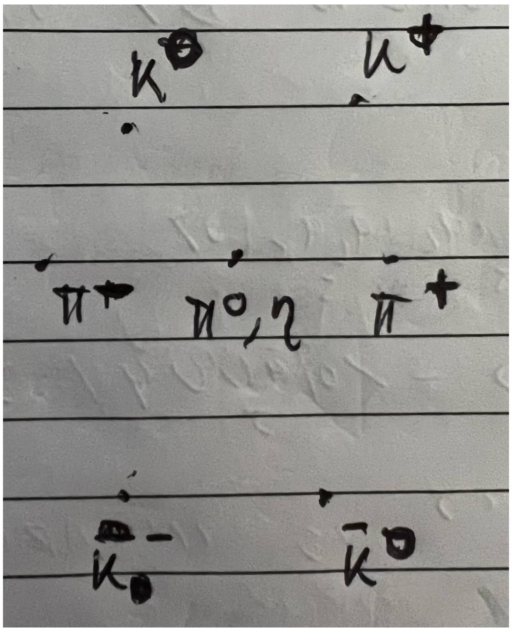

Figure 11: Meson Octet

Experimentally we find 9 particles with similar propertis, we associate them to the meson octet plus singlet. Their properties are listed in Tab. 1.

Particle	Quarks	Mass MeV	Lifetime (s)
$\pi^{\pm}$	$u\bar{d}$	140	10^{-8}
π^0	$u\bar{u} - d\bar{d}$	135	10^{-16}
$K^{\pm}$	$u\bar{s}$	494	10^{-8}
$K_0/\bar{K}_0$	$d\bar{s}$	498	$10^{-8}/10^{-11}$
η	$u\bar{u} + d\bar{d} - 2s\bar{s}$	548	10^{-19}
$\eta^{'}$	$u\bar{u} + d\bar{d} + s\bar{s}$	958	10^{-21}

Table 1: List of mesons in the meson octet and singlet.

There are 2 principal methods to explore non-perturbative effects of QCD. One is by using the so called Lattic QCD, where one tries to solve the equations of QCD on a lattice. The second is by using symmetries of the theory to build an effective lagrangian, this is called Chiral EFT. Using this approach, we get the following approximate relation for the meson masses, that try to explain the mass pattern

$$m_{ab}^2 = \frac{2\sigma}{f_\pi^2} Tr[\tau^a, m_a \tau^b] \tag{3.32}$$

where m_a is tha quark mass matrix, τ are the $SU(3)$ generators, f_π is the pion decay constant, not predicted by the theory, and σ is related to the value of the quark condensate. We get

$$m_\pi^2 = \frac{2\sigma}{f_\pi^2}(m_u + m_d) \tag{3.33}$$

$$m_{K^\pm} = \frac{2\sigma}{f_\pi^2}(m_u + m_s) \tag{3.34}$$

$$m_{K_0} = \frac{2\sigma}{f_\pi^2}(m_s + m_d) \tag{3.35}$$

$$m_\eta = \frac{2\sigma}{f_\pi^2}\frac{m_u + m_d + 4m_s}{3} \tag{3.36}$$

There are still several unresolved questions

1. Why the lifetime of the neutral pion is so much shorter than the ones of charged pions?
2. Why the neutral pion mass is not exactly the same as the ones of neutral pions, as predicted?
3. Why the singlet is much heavier than the mesons of the meson octet?
4. What about baryons? Are they goldstone bosons as well?

We will need to learn about anomalies to answer some of these questions, we can anticipate a few answer, however Regarding the barions, no they are not goldstone bosons, they would be massive also for massless quarks. We will not give a proof of this, but is related to the fact that they have non-zero baryon number. Regarding the mass of the singlet, the reason is that $\eta^{'}$ is not a Goldstone boson of the $U(1)_A$ symmetry, because such symmetry is not spontaneously broken as expected. in fact, we will see that such symmetry does not even exist, as is broken at the quantum level. This is called anomalous symmetry.

3.4 Quantum Anomalies

This subsection is here only for completeness, and will not be discussed at lecture, apart from the last part with the final result

We will see that $\eta^{'}$ s not a G.B. because $J^{\mu 5}$ is a current whose conservation is spoiled by quantum corrections, so we say it is anomalous. So the relative symmetry is not a symmetry of the action. No generators get spontaneusly broken, so there not G.B. originating. $\eta^{'}$ mass is generated by non-perturbative effects and would not go to zero for massless quarks.

3.4.1 Understanding anomalies

In a free fermion theory, the lagrangian of the free fermion, Ψ, can be divided in the one for the left handed component and the one for the right handed component:

$$\mathcal{L} = \bar{\Psi} i\partial\!\!\!/ \Psi = \bar{\Psi} i\partial\!\!\!/ \left(P_R + P_L\right) \Psi = \bar{\Psi}_R i\partial\!\!\!/ \Psi_R + \bar{\Psi}_L i\partial\!\!\!/ \Psi_L \tag{3.37}$$

In this case, one has 2 separete conserved currents, indicating that the number of left handed fermions and right handed fermions is separately conserved. The left and right handed currents can be recasted into

$$J^{\mu} = \Psi\gamma^{\mu}\Psi \tag{3.38}$$

$$J^{\mu 5} = \Psi\gamma^{\mu}\gamma^{5}\Psi \tag{3.39}$$

When adding interactions with a gauge boson, however, things change. We will be interested in expectation values of such currents in the presence of the external vector field, and in particular they divergency/conservation

$$\partial_{\mu}\langle A|J^{\mu}|A\rangle \tag{3.40}$$

$$\partial_{\mu}\langle A|J^{\mu 5}|A\rangle \tag{3.41}$$

If the current we are taking the expectation value of is a current associated to a gauge symmetry/boson, it has to be conserved. From the Ward identity, we expect the divergency of such expectation value to be zero, otherwise it would break gauge invariance.

if the current we are taking the expectation value of is associated to a global symmetry, a nonzero expectation value will just mean that such symmetry is spoiled at the quantum level.

Anomalies come from a specific class of diagrams, the "triangular diagrams" where one has fermions running in a loop, with 3 vertices for interactions. We start from the case where interactions are with a gauge boson. We know the QED case where the interaction have the form

$$eQ\gamma^{\mu} \tag{3.42}$$

and QCD where they have the form

$$g_s t^a \gamma^{\mu} \tag{3.43}$$

The QED current is therefore

$$J^{\mu}_{QED} = \sum_f Q_f \bar{f}\gamma^{\mu} f \tag{3.44}$$

and the QCD one is

$$J^{\mu,a}_{QCD} = \sum_f \bar{f}\gamma^\mu t^a f \tag{3.45}$$

There are both currents associated to local symmetries. regarding currents associated to global symemtries, we have the axial current of $U(1)_A$

$$J^{\mu 5} = \sum_q \bar{q}_R\gamma^\mu q_R - \sum_q \bar{q}_L\gamma^\mu q_L \tag{3.46}$$

and the $SU(2)_A$ or $SU(3)_A$ currents

$$J^{\mu 5a} = \bar{Q}_R\gamma^\mu t^a Q_R - \bar{Q}_L\gamma^\mu t^a Q_L \tag{3.47}$$

where now $Q = (u, d)$ for $SU(2)$ or $Q = (u, d, s)$ for $SU(3)$.

We can start by considering QED with just the electron. One can check that the relative triangle diagram dotted with momentum, so the divergency of the current, is zero. this happens because all vertices are $\propto \gamma^\mu$. We know that QED preserved parity. We now consider an $U(1)$ gauge theory with chiral interactions. For example we can consider a single right-handed fermion, with charge 1. The new diagram will be proportional to

$$\begin{aligned}\langle A|J^\mu|A\rangle &\propto Tr\left[\gamma^\mu P_R \not{k}\gamma^\nu P_R\not{k}\gamma^\rho P_R\not{k}\right]\epsilon_\nu\epsilon_\rho &(3.48)\\ &= Tr\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho P_R\not{k}\right]\epsilon_\nu\epsilon_\rho &(3.49)\end{aligned}$$

where $\not{k}$ just stands for some factor coming from numerators of fermion propagators, and we just used commutation relation and $P_R^2 = P_R$. Moreover, as we know that the divergency of the diagram with just γ^μ factors vanishes, it means that only the factor with γ^5 can have a nonzero divergency:

$$\partial_\mu\langle A|J^\mu|A\rangle \propto q_\mu Tr\left[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho\gamma^5\not{k}\right]\epsilon_\nu\epsilon_\rho = q^\mu M^\mu \neq 0 \tag{3.50}$$

we will see that indeed this diagram has nonzero value. This would indeed break gauge invariance of the theory, making it not consistent. if, however, we have multiple particles in the theory, with general charges Q_F then

$$\partial_\mu\langle A|J^\mu|A\rangle \propto q^\mu M^\mu \left(\sum_{f_R} Q^3_{F_R} - \sum_{f_L} Q^3_{F_L}\right) \tag{3.51}$$

For QED, one has an equal number of left and right handed particles, with equal charge, so that

$$\partial_\mu\langle A|J^\mu|A\rangle \propto q^\mu M^\mu \left(\sum_{f_R} Q^3_{F_R} - \sum_{f_L} Q^3_{F_L}\right) = q^\mu M^\mu \left(\sum_f Q^3_{F_R} - Q^3_{F_L}\right) = 0 \tag{3.52}$$

So, while the diagram is not zero, the overall group factor coming out of the sum over all fermions in all representations can be zero, making the current preserved.

Given the value of $q^\mu M^\mu$, we will see that the kind of anomalies that matter are

$$\begin{aligned}&local^3 &(3.53)\\ &global \times local^2 &(3.54)\end{aligned}$$

The first kind would spoil gauge invariance and make the theory inconsistent. The second kind would spoil the global current, making it not conserved. if the current was associated to a would-be goldstone boson, then that particle is not a goldstone boson due to the anomaly. In such case, the very same diagram allows the decay of such particle into a couple of gauge bosons.

3.4.2 Anomaly of Axial current

We now want to calculate the divergence of the $J^{\mu 5}$ current. it is given by 2 diagrams in Fig. 12. We can neglect the fermion masses.

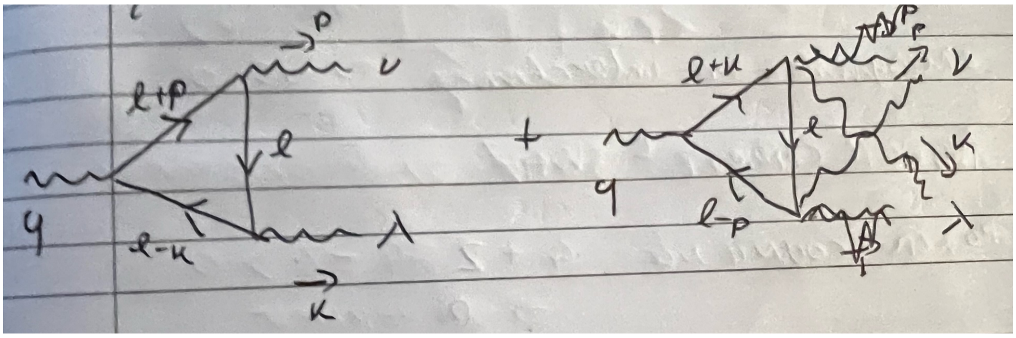

Figure 12: Diagrams contributing to anomaly

$$iJ^{\mu} = -(-ie)^2 \int \frac{d^4l}{(2\pi)^4} Tr[\gamma^{\mu}\gamma^5 \frac{i(\not{l}-\not{k})}{(l-k)^2}\gamma^{\lambda}\frac{i\not{l}}{l^2}\gamma^{\nu}\frac{i(\not{l}+\not{p})}{(l+p)^2}] + (p,\nu) \leftrightarrow k,\lambda) \tag{3.55}$$

$$\partial_{\mu}J^{5,\mu} = iq_{\mu}J^{\mu} \tag{3.56}$$

$$\not{q}\gamma^5 = (\not{l}+\not{p}-(\not{l}-\not{k})\gamma^5 = (\not{l}+\not{p})\gamma^5 + \gamma^5(\not{l}-\not{k}) \tag{3.57}$$

$$\partial_{\mu}J^{5,\mu} = e^2 \int \frac{d^4l}{(2\pi)^4} Tr[\gamma^5 \frac{\not{l}-\not{k}}{(l-k)^2}\gamma^{\lambda}\frac{\not{l}}{l^2}\gamma^{\nu} + \gamma^5\gamma^{\lambda}\frac{\not{l}}{l^2}\gamma^{\nu}\frac{\not{l}+\not{p}}{(l+p)^2}] + (p,\nu) \leftrightarrow k,\lambda) \tag{3.58}$$

$$= e^2 \int \frac{d^4l}{(2\pi)^4} Tr[\gamma^5 \frac{\not{l}-\not{k}}{(l-k)^2}\gamma^{\lambda}\frac{\not{l}}{l^2}\gamma^{\nu} - \gamma^5\frac{\not{l}}{l^2}\gamma^{\nu}\frac{\not{l}+\not{p}}{(l+p)^2}\gamma^{\lambda}] + (p,\nu) \leftrightarrow k,\lambda) \tag{3.59}$$

Now, one would be tempted to say that the result is zero, cause if I shift the first part by

$$l^{\mu} \to l^{\mu} + k^{\mu} \tag{3.60}$$

one gets a result that is asymmetric in the interchange, and therefore the 2 diagrams would cancel. But we can't make that shift! Because the integral is badly divergent. By power counting $4+2-4 = 2$ is quadratically divergent. The shift might not be allowed by dimensional regularization in d dimensions. To go to d dimensions, we need a d dimensional definition of γ^5, that is an object that is strictly 4-dimensional. We use

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.61}$$

$$\{\gamma^5, \gamma^i\} = 0, \quad i = 0,1,2,3 \tag{3.62}$$

$$[\gamma^5, \gamma^i] = 0, \quad i \geq 4 \tag{3.63}$$

We decompose l as the sum of the first 4 dimensions $l_{||}$ and the additional dimensions $l_{\perp}$.

$$l = l_{||} + l_{\perp} \tag{3.64}$$

then

$$\not{q}\gamma^5 = (\not{l}+\not{p}-(\not{l}-\not{k})\gamma^5 = (\not{l}+\not{p})\gamma^5 + \gamma^5(\not{l}-\not{k}) - 2\gamma^5 l_{\perp} \tag{3.65}$$

and the previous result is modified accordingly. Now we can perform the shift, and the additional term survives

$$= e^2 \int \frac{d^4l}{(2\pi)^4} Tr[-2\gamma^5 \not{l}_\perp \frac{\not{l} - \not{k}}{(l-k)^2} \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu \frac{\not{l} + \not{p}}{(l+p)^2}] + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.66}$$

Now, as usual we combine denominators and sfift the momentum

$$l \to l + xk - yp \tag{3.67}$$

because of the γ^5 factor, we need 4 different (not parallel) vectors in the first 4 dimensions to get a nonzero trace. There is only one surviving term

$$= e^2 \int \frac{d^4l}{(2\pi)^4} \frac{Tr[-2\gamma^5 \not{l}_\perp (-) \not{k} \gamma^\lambda \not{l}_\perp \gamma^\nu \not{p}]}{(l^2 - \Delta)^3} + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.68}$$

now, $\not{l}_\perp$ commutes, so

$$= e^2 \int \frac{d^4l}{(2\pi)^4} \frac{Tr[-2\gamma^5 \not{l}_\perp \not{l}_\perp (-) \not{k} \gamma^\lambda \gamma^\nu \not{p}]}{(l^2 - \Delta)^3} + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.69}$$

$$= 2e^2 Tr[\gamma^5 \not{k} \gamma^\lambda \gamma^\nu \not{p}] \int \frac{d^4l}{(2\pi)^4} \frac{l_\perp^2}{(l^2 - \Delta)^3} + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.70}$$

$$= 2e^2 \frac{d-4}{d} Tr[\gamma^5 \not{k} \gamma^\lambda \gamma^\nu \not{p}] \int \frac{d^4l}{(2\pi)^4} \frac{l^2}{(l^2 - \Delta)^3} + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.71}$$

$$= 2e^2 \frac{d-4}{d} Tr[\gamma^5 \not{k} \gamma^\lambda \gamma^\nu \not{p}] \left(\frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{2} \frac{1}{\Delta^{2-\frac{d}{2}}} \right) + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.72}$$

$$= -\frac{ie^2}{(4\pi)^{d/2}} Tr[\gamma^5 \not{k} \gamma^\lambda \gamma^\nu \not{p}] \frac{1}{\Delta^{2-\frac{d}{2}}} + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.73}$$

$$= -\frac{ie^2}{(4\pi)^2} i4\epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta + (p, \nu) \overset{(}{\leftrightarrow} k, \lambda) \tag{3.74}$$

$$= \frac{e^2}{2\pi^2} \epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta \tag{3.75}$$

$$= -\frac{e^2}{16\pi^2} \langle p, k | \epsilon^{\alpha\lambda\beta\nu} F^{\alpha\nu} F_{\beta\lambda} | 0 \rangle \tag{3.76}$$

This is called the Alder-Bell-Jackiw anomaly.

For any 3 symmetries of the lagrangian, we can associate the relative anomaly given by triangle diagrams with fermion loops. This is something we want to keep in mind for later on, when we will check that the SM gauge symmetries are not broken by anomalies. We can therefore calculate the $U(1)_A \times G^2$ anomaly for any symmetry group G.

One can perform the same computation in the case of 2 external gluons, rather than photons. The result is the same as for QED, with the addition of a group factor (and summing up over flavours)

$$\partial_\mu J^{5,\mu} = -\frac{g_3^2}{16\pi^2} Tr[t^a t^b] Tr[\mathbb{1}_F] G_a^{\mu\nu} G_{b,\mu\nu} = -\frac{g_3^2 N_f}{32\pi^2} G_a^{\mu\nu} G_{a,\mu\nu} \tag{3.77}$$

3.5 Consequences of Anomaly on Global symmetries

We now check if the currents $J^{\mu 5}$ and $J^{\mu 53}$ are spoiled by coupling the quarks to gluons. So the relevant diagrams are

$$J^{\mu 5} \times SU(3)_c^2 \tag{3.78}$$
$$J^{\mu 53} \times SU(3)_c^2 \tag{3.79}$$

As the single diagram is not zero, we need to just check the group factor. For $J^{\mu 5}$, every right handed particle has a "charge" $Q_R = +1/2$, while every left handed particle has a "charge" $Q_L = -1/2$. The overall factor is

$$\sum_f Tr[t^a t^b] \sum_q (Q_R - Q_L) = N_f \frac{\delta^{ab}}{2} \times 1 = \frac{N_F}{2} \delta^{ab} \neq 0 \tag{3.80}$$

as we had just seen in the previous section. So the current has a QCD anomaly. his means $\eta^{'}$ is not a GB. As a first consequence, we expect therefore that the mass of such particle depends completely on the non-perturbative scale Λ_{QCD}, and not on the quark masses, so it should be considerably higher. This replies to our question 3.

For the other current, we can use the vector form of quarks, and the matrix for t^3 is the generator. The left-handed particle however transform with the inverse matrix, so get a factor $-t^3$ the group factor is

$$Tr[t^a t^b] \left(Tr_R \left[t^3 \right] - Tr_L \left[-t^3 \right] \right) = Tr[t^a t^b] Tr \left[t^3 \right] = 0 \tag{3.81}$$

So the neutral pion has no QCD anomaly - it is a GB!

Now we add interactions with the photon and we turn to anomalies with the QED photon, so

$$J^{\mu 5} \times U(1)_{em}^2 \tag{3.82}$$
$$J^{\mu 53} \times U(1)_{em}^2 \tag{3.83}$$

The 2 group factors in this case are

$$N_c \sum_f Q_{f,A,R} Q_{f,R}^2 - Q_{f,A,L} Q_{f,L}^2 = \frac{N_c}{2} \sum_f Q_{f,R}^2 - (-1) Q_{f,L}^2 = N_c \sum_f Q_f^2 \tag{3.84}$$

$$N_c \left(Tr_R \left[t^3 Q_R^2 \right] - Tr_L \left[-t^3 Q_L^2 \right] \right) = 2 N_c Tr \left[t^3 Q^2 \right] = N_c \frac{1}{2} \left(Q_u^2 - Q_d^2 \right) = \frac{N_c}{6} \tag{3.85}$$

Both coefficients are non-zero, so this means both currents develop an anomaly with $U(1)_{em}$ that allows the relative particle to decay to 2 photons. . The existence of the anomaly with the EM gauge group generates a non-zero coefficient for the operator

$$\pi^0 \epsilon^{\alpha\lambda\beta\nu} F^{\alpha\nu} F_{\beta\lambda} = \pi^0 F^{\mu\nu} \tilde{F}_{\mu\nu} \tag{3.86}$$

This comes one summing up over all flavours

$$\partial_\mu J^{5a,\mu} = -\frac{e^2}{16\pi^2} Tr[t^a Q^2] Tr[\mathbb{1}_c] F^{\mu\nu} F_{\mu\nu} = -\frac{e^2}{32\pi^2} F^{\mu\nu} F_{\mu\nu} \tag{3.87}$$

Such operator allows the neutral pion to decay by EM interaction, so

$$\mathcal{M}_{\pi^0} \propto \frac{e^2}{16\pi^2} \tag{3.88}$$

The charged pions, instead, do not suffer from the $U(1)_A \times U(1)^2_{em}$ anomaly. They are therefore stable under the EM interaction, and can only decay through the weak interaction, meaning that their decay rate will be

$$\mathcal{M}_{\pi^+} \propto e^2 \frac{m_\pi^2}{M_W^2} \sim 10^{-6} e^2 \sim 2.5 \times 10^{-4} \mathcal{M}_{\pi^0} \tag{3.89}$$

So as a first rough estimate one would get, for the neutral pion

$$\Gamma_{\pi^\pm} \sim 10^{-7} \Gamma_{\pi^0} \tag{3.90}$$

This factor is not precise, as that requires to perform a full calculation of the matrix element, but it is however of the right order of magnitude and approximately explains the relative ratio of the decay rates. This answers our question 1.

Regarding question 2, we can notice that we have defined a flavour symmetry were all quark flavours belong to the same representation. particles living in the same representation should all have the same quantum numbers, w.r.t the symmetries of the lagrangian. This is correct if one includes only QCD interaction in the lagrangian. If one tries to account for EM interactions in the lagrangian, however, the flavour symmetry breaks, as u and d have different electric charge. It is therefore expected that neutral and charged pion cannot have exactly the same mass, with the difference being in part due to EM interactions, and in part coming already in the case of exact flavour symmetry, as using the relation 3.32 one gets a mixed term $m^2_{38} = m^2_{\pi^0 \eta}$ that causes a mixing between them and changes the value of the mass eigenvalues by $\mathcal{O}(\frac{2\sigma}{f_\pi^2} \frac{3(m_u+m_d)}{4m_s+m_u+m_d})$. This is our reply for question 2.

There are other questions that one could raise by looking at Tab.1. The most interesting one is related to the neutral kaons $k_0, \bar{K}_0$, their mass and their decay times. This is an interesting topic related to CP breaking in the SM, but we will not cover this in the course.

References