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The Standard Model Lagrangian (Lecture 1)

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Abstract.

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1.1 Covariant Derivarive

1.1.1 Abelian Case

We consider a fermion field Ψ with electric charge Q. We will say that Ψ belongs to the Q representation of $U(1)$, and it will transform as

$$\Psi \to e^{ieQ\alpha(x,t)}\Psi \tag{1.1}$$

An arbitrary phase shit that depends on the space-time point is not problematic for $\bar{\Psi}\Psi$ terms, as

$$\bar{\Psi}\Psi \to \bar{\Psi}\Psi \tag{1.2}$$

The terms that have a problem are the ones involving a derivative. The derivative along a direction n^μ is

$$n^\mu \partial_\mu \Psi = \lim_{\epsilon \to 0} \frac{\Psi(x^\mu + \epsilon n^\mu) - \Psi(x^\mu)}{\epsilon} \tag{1.3}$$

This is a difference of Ψ at DIFFERENT space-time points, that means the difference of fields that transform differently under the phase shift, as the first one will acquire a phase $e\alpha(x^\mu + \epsilon n^\mu)$, while the second will acquire a phase $e\alpha(x^\mu)$. To obtain an object that transforms correctly, we need to define a transport operator

$$U(y,x) \to e^{ieQ\alpha(y)}U(y,x)e^{-ieQ\alpha(x)} \tag{1.4}$$

$$U(x,x) = 1 \tag{1.5}$$

$$\|U\| = 1 \tag{1.6}$$

where the last equation means that U can (should) be takes as a pure phase. Following what we have seen so far, we can see that a good choice can be

$$U(y,x) = e^{ieQ\int_x^y A_\mu dl^\mu} \tag{1.7}$$

Using this operator we get that

$$U(x^\mu + \epsilon n^\mu, x^\mu)\Psi(x^\mu) \tag{1.8}$$

transforms in the same way as

$$\Psi(x^\mu + \epsilon n^\mu), \tag{1.9}$$

so we can define

$$\begin{aligned}
n^\mu D_\mu \Psi &= \lim_{\epsilon\to 0} \frac{\Psi(x^\mu + \epsilon n^\mu) - U(x^\mu + \epsilon n^\mu, x^\mu)\Psi(x^\mu)}{\epsilon} && (1.10)\\
&= \lim_{\epsilon\to 0} \frac{\Psi(x^\mu + \epsilon n^\mu) - e^{ieQ\epsilon n^\mu A_\mu}\Psi(x^\mu)}{\epsilon} && (1.11)\\
&= \lim_{\epsilon\to 0} \frac{\Psi(x^\mu) + \epsilon n^\mu \partial_\mu \Psi(x^\mu) - (1 + ieQ\epsilon n^\mu A_\mu)\Psi(x^\mu)}{\epsilon} && (1.12)\\
&= n^\mu \left(\partial_\mu - ieQA_\mu\right)\Psi(x^\mu) && (1.13)
\end{aligned}$$

so

$$D_\mu = \partial_\mu - ieQA_\mu \tag{1.14}$$

Exercise Imposing that this object transforms like Ψ, find the transformation law for A_μ.

1.1.2 Non-Abelian Case

The field Ψ now belongs to a representation space of $SU(N)$, for example the fundamental one, and will have a group index a (in other words, it can be seen as a column vector).

The operator U still needs to be a unitary operator, belonging to the non-abelian gauge group, so:

$$U(y,x) = e^{ig\int_x^y dl^\mu (A_\mu^a t^a)}, \tag{1.15}$$

where A_μ^a are a set of fields belonging to the adjoint representation of $SU(N)$ and t^a are the $SU(N)$ generators.

All the previous steps still apply, as we have assumed nowhere that U was a c-number. So

$$D_\mu = \partial_\mu - igA_\mu^a t^a \tag{1.16}$$

Exercise What are the transformation laws for A_μ^a?

1.2 Field Strength Tensor

Given that D_μ has nice transformation properties, we want to use this object to contract other operators that still retain such nice transformation properties. In the abelian case, we note that

$$\begin{aligned} D_\mu &\rightarrow U D_\mu U^\dagger & (1.17)\\ U(x) &= e^{ie\alpha(x)} & (1.18)\\ [D_\mu, D_\nu] &= -ieF_{\mu\nu} & (1.19)\\ F_{\mu\nu} &\rightarrow U F_{\mu\nu} U^\dagger = F_{\mu\nu} & (1.20) \end{aligned}$$

Similarly, we can do the same for the non abelian case

$$\begin{aligned} D_\mu &\rightarrow U D_\mu U^\dagger & (1.21)\\ U(x) &= e^{ig\alpha^a(x)t^a} & (1.22)\\ [D_\mu, D_\nu] &= -igF_{\mu\nu} & (1.23)\\ F_{\mu\nu} &\rightarrow U F_{\mu\nu} U^\dagger & (1.24) \end{aligned}$$

however this time $F_{\mu\nu}$ will not be invariant, but will just transform "in the right way" under gauge transformations.

1.3 Kinetic Terms

In the Abelian case, given the invariance of $F_{\mu\nu}$, together with the fact that it was a c-number, one was taking

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.25}$$

In the non-Abelian case, $F_{\mu\nu}$ is not invariant, and is not a c-number, but a matrix. We need to construct an an invariant object that is also a c-number out of it. We note that

$$F_{\mu\nu} F^{\mu\nu} \rightarrow U F_{\mu\nu} F^{\mu\nu} U^\dagger \tag{1.26}$$

So, the only possible choice is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} Tr[F_{\mu\nu} F^{\mu\nu}] = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} & (1.27)\\ F^a_{\mu\nu} &= 2Tr[F_{\mu\nu} t^a] = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - igTr[[A^b_\mu \frac{t^b}{2}, A^c_\nu \frac{t^c}{2}] t^a] & (1.28)\\ &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu & (1.29) \end{aligned}$$

Exercise Use the transport operator U to find $F_{\mu\nu}$.

1.4 Color

Gell-mann matrices can be associated to color carried by gluons, and color is conserved at each vertex.

Exercise Find all Gluon-matries correspondences, as in Fig. 1.

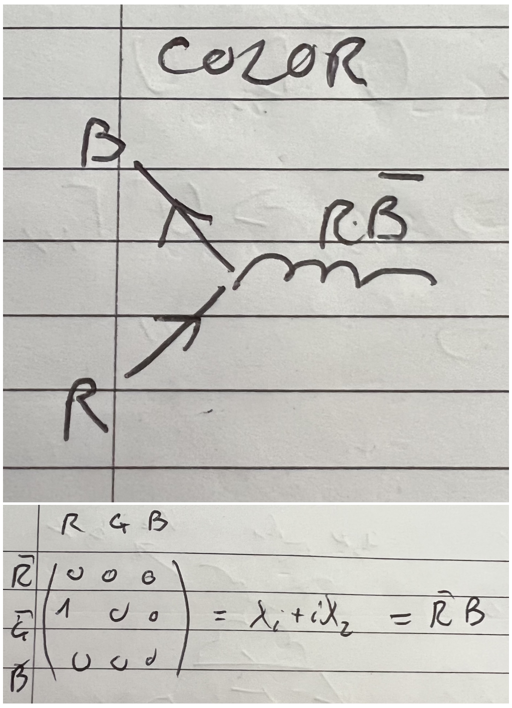

Figure 1: Example of gluon-matrix correspondence

Gluon-Gluon vertices
The arise from the 3 and 4 gluon terms of

$$(F^a_{\mu\nu})^2 = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu\right)^2 \tag{1.30}$$

For the 4-gluon vertex we get

$$\mathcal{L} = g^2 f_{abc} A^b_\mu A^c_\nu f_{ade} A^{\mu,d} A^{\mu,e} \tag{1.31}$$

so the feynman rule reads

$$(-i) g^2 f_{abc} f_{ade} \left(g_{\mu\rho} g_{\nu\sigma} + permutations\right) \tag{1.32}$$

Exercise Find the F.R. for the ggg vertex. Note: beware of the sign of the momenta! Use the expression for A_μ in terms of creation and annihilation operators to work them out correctly! Take all the momenta to be ingoing.

1.5 Gauge Group and Yang-Mills Lagrangian

The first thing one needs to introduce when defining a particle physics model is the gauge group. The SM gauge group is

$$SU(3)_c \times SU(2)_L \times U(1)_Y \tag{1.33}$$

F	$SU(3)$	$SU(2)$	$U(1)$
Q_L	3	2	1/6
u_R	3	1	2/3
d_R	3	1	$-1/3$
L_L	1	2	$-1/2$
e_R	1	1	-1

Table 1: Standard Model Fermion field content

Following our section on Gauge invariance for Abelian and non-Abelian gauge groups, the relative kinetic terms for the gauge bosons is

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}B^{mu\nu} - \frac{1}{2}Tr[W_{\mu\nu}W^{\mu\nu}] - \frac{1}{2}Tr[G_{\mu\nu}G^{\mu\nu}] \tag{1.34}$$

$$= -\frac{1}{4}B_{\mu\nu}B^{mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{\mu\nu,a} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} \tag{1.35}$$

2 Fermion (Dirac) terms

2.1 Fermion Content and Fermion kinetic terms

The next step is to define the field content of the theory. We start from the fermion content. For the SM, the fermionic field content is, with each field assigned to a specific gauge group representation. This is done for the SM in Tab. 1. This uniquely defines the covariant derivative for each field. The kinetic term of the lagrangian is exacily obtained as

$$\mathcal{L} = \bar{Q_L}i\not{D}Q_L + \bar{u_R}i\not{D}u_R + \bar{d_R}i\not{D}d_R + \bar{L_L}i\not{D}L_L + \bar{e_R}i\not{D}e_R \tag{2.1}$$

Note that left and right fields are separated here. This is necessary because parity is not conserved by weak interactions, as discovered by Madame Wu in 1956. However, this raises up some problems, as we will see in the next sections.

References