Topological states of matter: topological order vs SPT phases

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1 Topological states of matter

1.1 Gapful phases of quantum matter

Suppose we have a many-body local Hamiltonian $H(\vec{\lambda})$ which depends on the set of parameters $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m)$. Its Schrödinger equation reads

$$H(\vec{\lambda})\psi_n(\vec{\lambda}) = E_n(\vec{\lambda})\psi_n(\vec{\lambda}). \tag{1.1}$$

Among the solutions to this equation, there is the one with the smallest possible $E = E_0$. We call it the ground state. The next one up is $E_1 > E_0$, $E_1 < E_n$, $n = 2, 3, \dots$. This is the first excited state.

$$\Delta(\vec{\lambda}) = E_1(\vec{\lambda}) - E_0(\vec{\lambda}) \tag{1.2}$$

is the gap.

The gap is typically not zero in a variety of systems with a finite number of particles. As the number of particles is taken to infinity (the thermodynamic limit), the gap often shrinks to zero. In other cases, it remains finite.

The region in the space of $\vec{\lambda}$ where $\Delta > 0$ even in the limit of an infinite number of particles is referred to as gapful phase of quantum matter.

Two distinct gapful phases are those domains which are separated by phase transition where the gap Δ closes. That is, saying that $\vec{\lambda}_i$ belongs to one phase and $\vec{\lambda}_f$ belong to another phase, is equivalent to saying that, first of all $\Delta(\vec{\lambda}_i) > 0$, $\Delta(\vec{\lambda}_f) > 0$, and second, for any continuous function $\vec{\lambda}(s)$, $\vec{\lambda}(0) = \lambda_i$, $\vec{\lambda}(1) = \lambda_f$, there will be some $0 < s < 1$ such that $\Delta(\vec{\lambda}(s)) = 0$.

1.2 Symmetry breaking gapful phases of quantum matter

According to Landau's theory of phase transitions, distinct phases of matter break symmetries spontaneously in different ways. Consider for example the famous transverse field Ising model. Its Hamiltonian is

$$H = -\beta \sum_{n=1}^{L-1} \sigma_n^z \sigma_{n+1}^z - \gamma \sum_n \sigma_n^x. \tag{1.3}$$

Here σ_n are Pauli matrices acting on the n-th spin-1/2 particle.

If $\beta = 0$, the ground state of this model is easy to find. It maximizes each σ_n^x for each n, so it is a product of eigenstates of σ^x with the maximum eigenvalue of $+1$. In other words,

$$|\Psi_{\rm GS}\rangle_{\beta=0} = \prod_{n=1}^{N} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n . \tag{1.4}$$

Excitations correspond to flipping one spin and cost energy

$$\Delta = 2\gamma. \tag{1.5}$$

On the other hand, if $\gamma = 0$, then there are actually two ground states of this problem:

$$\left|\Psi_{\rm GS}^{(1)}\right\rangle_{\gamma=0} = \prod_{n=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n , \quad \left|\Psi_{\rm GS}^{(2)}\right\rangle_{\gamma=0} = \prod_{n=1}^{N} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_n , \tag{1.6}$$

The lowest energy excitation is a domain wall where spins to the left of a given n point up, and they point down to the right of this n. The energy cost of this is

$$\Delta = 2\beta. \tag{1.7}$$

Clearly in these two limits these are gapful phases of this Hamiltonian. We can now argue that these are two distinct phases. That is, as the ratio

$$\lambda = \beta/\gamma \tag{1.8}$$

is tuned from 0 to ∞, there must a phase transition between the phases at some critical λ.

Indeed, the Hamiltonian commutes with the symmetry operation

$$P = \prod_{n=1}^{L} \sigma_n^x . \tag{1.9}$$

$$HP - PH = 0. \tag{1.10}$$

Yet the ground states of the Hamiltonian in case when $\gamma = 0$, or equivalently $\lambda = \infty$, breaks the symmetry as

$$P\left|\Psi_{\rm GS}^{(1)}\right\rangle_{\gamma=0} = \left|\Psi_{\rm GS}^{(2)}\right\rangle_{\gamma=0} , \tag{1.11}$$

instead of being invariant under the action of the symmetry. This is called spontaneous symmetry breaking, since the Hamiltonian is symmetric while the ground state is not.

At the same time, when $\beta = 0$, or equivalently $\lambda = 0$, the ground state does not break symmetry, or

$$P \left|\Psi_{\mathrm{GS}}\right\rangle_{\beta=0} = \left|\Psi_{\mathrm{GS}}\right\rangle_{\beta=0}, \tag{1.12}$$

as can be explicitly checked.

Therefore, as λ is varied from 0 to ∞, there must be some special value of λ where symmetry gets broken. This is the value where the phase transition occurs. Therefore, even without solving this model we know it should have two distinct phases separated by a phase transition.

This model has the advantage of being solvable exactly. From the exact solution it is known that $\Delta = 2\left|\beta - \gamma\right|$, and that the transition occurs when $\beta = \gamma$, or $\lambda = 1$.

1.3 Topological phases of matter

It is now known that distinct phases of quantum matter can occur even when spontaneous symmetry breaking does not happen. Such phases are called topological phases.

More precisely, we define: **topological phases** are gapful states which do not spontaneously break any symmetries and which cannot be converted into each other without closing the gap (thereby going through a phase transition).

Furthermore, we will say that if topological phases can never be converted into each other, these will be states with **topological order**. If, on the other hand, these phases can be converted into each other but only if terms are added to the Hamiltonian which will break some symmetries of the Hamiltonian (and so as long as Hamiltonian preserves certain symmetries as $\vec{\lambda}$ changes, they cannot be converted into each other without closing the gap), then we will call these **symmetry protected topological phases**, or SPT phases.

Topologically ordered phases and symmetry protected phases are the two main classes of topological states.

We note that the Hamiltonians of these phases must be local (that is, would not allow for long range interactions). Indeed, if the Hamiltonian can be arbitrary we can always construct a fake Hamiltonian for any set of $\vec{\lambda}$-dependent "ground state" wave functions $\psi_0(\vec{\lambda})$ which is equal to the minus projection operator into this wave function. Such a Hamiltonian has the ground state "energy" of -1 and the gap $\Delta = 1$ which never turns to zero at any $\vec{\lambda}$. For these Hamiltonians there could only be one phase of matter and the gap never closes.

However, such a Hamiltonian is typically not local. Local Hamiltonians do allow for distinct phases which cannot be converted into each other without closing the gap.

2 Topologically ordered phases

2.1 Quantum entanglement

The main feature of the topologically ordered phases is that they possess long ranged entanglement. Entanglement is a property of quantum particles. Suppose we have two spin-1/2 particles, with the phase function given by $\psi(\sigma_1, \sigma_2)$, with σ_1 and σ_2 taking values 1 and 2, corresponding to spin-up and spin-down states. We can use it to construct the reduced density matrix of one particle with the other one summed over:

$$\rho(\sigma_1, \sigma_1') = \sum_{\sigma_2} \psi(\sigma_1, \sigma_2)\, \psi^*(\sigma_1', \sigma_2). \tag{2.13}$$

With the help of this reduced density matrix, we can defined the entanglement entropy, the measure of quantum entanglement of two spins. It is defined as

$$S = -\mathrm{tr}\, \left[\rho \ln \rho\right]. \tag{2.14}$$

It is easy to verify that if the spins are not entangled, in other words, if

$$\psi(\sigma_1, \sigma_2) = \psi_1(\sigma_1)\psi_2(\sigma_2), \tag{2.15}$$

then

$$\rho = \psi_1(\sigma_1)\psi_1^*(\sigma_1'). \tag{2.16}$$

This is due to $\sum_{\sigma_2=1,2} \left|\psi_2(\sigma_2)\right|^2 = 1$ as the wave function must be normalized.

Such ρ has two eigenvalues, 1 and 0. Since

$$S = -\sum_n \lambda_n \ln \lambda_n, \tag{2.17}$$

where λ_n are eigenvalues of the density matrix, then $S = 0$.

On the other hand, if the two spins are entangled, for example, if they together form a spin-0 particle with the wave function

$$\psi(\sigma_1, \sigma_2) = \frac{1}{\sqrt{2}} \left(\delta_{\sigma_1,1}\delta_{\sigma_2,2} - \delta_{\sigma_1,2}\delta_{\sigma_2,1} \right), \tag{2.18}$$

then the density matrix is

$$\rho(\sigma_1, \sigma_1') = \frac{1}{2}\delta_{\sigma_1,\sigma_1'}. \tag{2.19}$$

Its eigenvalues are $\lambda_1 = \lambda_2 = 1/2$, and

$$S = \ln 2. \tag{2.20}$$

Two spins are maximally entangled if their entanglement entropy is $\ln 2$, and they are not entangled if it is 0. One can show that entanglement entropy of two spins can never be negative and can never exceed $\ln 2$.

2.2 Entanglement in topologically ordered phases

Suppose we have a Hamiltonian $H(\vec{\lambda}_i)$ for some $\vec{\lambda}_i$ such that its ground state $\psi_0(\vec{\lambda}_i)$ is a product state (its wave function is a product of wave function of individual particles/spins in points in space). Suppose at some other $\vec{\lambda}_f$ the ground state is no longer a product state but belongs to the same phase. We can argue that $\psi_0(\vec{\lambda}_f)$ can only have short range entanglement (no entanglement among spins/particles which are far away from each other).

Indeed, by definition given earlier we can find $\vec{\lambda}(t)$, $\vec{\lambda}(0) = \vec{\lambda}_i$, $\vec{\lambda}(T) = \vec{\lambda}_f$ such that the gap never closes for all $H(\vec{\lambda}(t))$ with $0 \le t \le T$. If the gap never closes we can argue that we can find $\psi_0(\vec{\lambda}(t))$ by solving the Schrodinger equation

$$i\frac{\partial \psi}{\partial t} = H(\vec{\lambda}(t))\psi, \tag{2.21}$$

with the initial conditions

$$\psi(0) = \psi_0(\vec{\lambda}_i). \tag{2.22}$$

This is guaranteed by the adiabatic theorem of quantum mechanics as long as

$$\frac{d\lambda}{dt} \ll \Delta(\vec{\lambda}(t)). \tag{2.23}$$

It follows from here that $T \sim 1/\Delta$. We can split the interval T into T/ϵ steps of time ϵ each and write

$$\psi_0(\vec{\lambda}(T)) = \prod_{j=0}^{T/\epsilon} e^{-i\epsilon H(\vec{\lambda}(j\epsilon))} \psi_0(\vec{\lambda}(0)) \approx \prod_{j=0}^{T/\epsilon} \left[1 - i\epsilon H(\vec{\lambda}(j\epsilon))\right] \psi_0(\vec{\lambda}(0)). \tag{2.24}$$

This is the so-called Trotter decomposition of the evolution operator. Each term in the square brackets can only entangle two nearby spins/particles, since the Hamiltonians are local. Altogether there are T/ϵ terms which can entangle spins some distance away from each other, but since $T \sim 1/\Delta$ remains finite as the system size is taken to infinity, these terms cannot entangle very far away spins.

On the other hand, if we require that there is a phase transition as we go from $\vec{\lambda}_i$ to $\vec{\lambda}_f$, that means Δ turns to a small value (which becomes smaller as the system size increases) somewhere as $\vec{\lambda}$ is tuned between these values. Then T has to be very large as the system size gets large, and the Trotter decomposition ends up entangling spins/particles which are far away from each other.

Therefore, we see that states with topological order must be long ranged entangled. Furthermore, different topologically ordered phases must be entangled in different ways.

2.3 Topologically ordered phases: examples

It is believed that there cannot be any topologically ordered phases in 1D.

In 2D the famous topologically ordered phases are seen in fractional quantum Hall effect. Other examples are certain gapful spin liquids. Of those, a famous example is given by the exactly solvable model of a gapful spin liquid called Kitaev's toric code. All the topologically ordered phases in 2D have long ranged entanglement characterized by the entanglement entropy in the following way.

Take a two dimensional system and take a domain A inside it, with the rest of the system belonging to domain B. Construct the reduced density matrix of A with respect to B from the ground state wave function, and calculate the entanglement entropy of A with respect to B. It can be shown that it is equal to

$$S \sim c\ell - \gamma. \tag{2.25}$$

Here ℓ is the length of the boundary separating A from B, c is some constant, and γ is the famous constant whose value characterizes the topological order of the phase. In Kitaev's toric code, $\gamma = \ln 2$ as follows from its exact solution.

One can argue that nonzero γ reflects long ranged entanglement in the system.

Two dimensional systems with topological order have excitations which are neither bosons nor fermions, but rather generalizes the notion of statistics. They also have fractional charge (with respect to the charge of the particles which formed the interacting many body system which found itself in the topological phase). It is clear that noninteracting systems cannot have topological order.

In 3D until recently it was argued that topologically ordered phases were direct generalizations of those found in 2D. Very recently however a new type of topologically ordered phases in 3D were discovered, called fracton phases. They appear to be the new frontier in the study of topologically ordered phases.

3 Symmetry protected topological phases of matter

3.1 Topological insulators

Unlike states with topological order, noninteracting particles can form symmetry protected states. Those received the names of topological insulators (and superconductors, since BdG Hamiltonians describing superconductors can be thought of as non-interacting).

Those generally can be written as

$$H = \sum_{nm} \mathcal{H}_{nm} a^\dagger_n a_m, \tag{3.26}$$

where $a_n^\dagger$ and a_n are creation and annihilation operators of some fermions (electrons or cold atoms) at a site n of a lattice.

The study of such Hamiltonians reduces to study of the energy spectrum of $\mathcal{H}$. Those spectra typically form bands. If a band is fully filled by fermions, with bands above it empty, the system has a gap. An SPT phase would correspond to a situation where different types of filled bands cannot be converted into each other without merging them first (which would require in closing of the gap), as long as certain symmetries are preserved. Different symmetries result in the different SPT phases.

In the theory of topological insulators it was found that these bands can be labelled by certain integer numbers, topological invariants. Bands with a particular topological invariant cannot be converted into a band with another value of this invariant without merging it with other bands first, requiring a phase transition.

Nontrivial SPT phases always have gapless edge states at their boundaries. This can be contrasted with the states with topological order which do not always have gapless edge states.

3.2 Example of the SPT phase: SSH model

An example useful for further discussions is the Su-Schrieffer-Heeger model. This is a one dimensional model of spinless noninteracting fermions hopping on a lattice with varying hopping matrix element. It has a symmetry usually referred to as chiral symmetry which protects its distinct SPT phases. In addition, it is also usually taken to be time-reversal invariant (has a real Hamiltonian).

The model has the Hamiltonian

$$H = \sum_{\alpha=1}^{N} \sum_n \left(t + (-1)^n \delta t\right) a_n^{\dagger\,(\alpha)} a_{n+1}^{(\alpha)} + \text{h.c.}. \tag{3.27}$$

Here as usual $a_n^{\dagger\,(\alpha)}$ and $a_n^{(\alpha)}$ are creation and annihilation operators of fermions of species α on the site n. Those obey the usual anticommutation relations

$$a_n^{\dagger\,(\alpha)} a_m^{\dagger\,(\beta)} + a_m^{\dagger\,(\beta)} a_n^{\dagger\,(\alpha)} = 0, \;\; a_n{}^{(\alpha)} a_m{}^{(\beta)} + a_m{}^{(\beta)} a_n{}^{(\alpha)} = 0, \tag{3.28}$$
$$a_n^{\dagger\,(\alpha)} a_m{}^{(\beta)} + a_m{}^{(\beta)} a_n^{\dagger\,(\alpha)} = \delta^{\alpha\beta} \delta_{nm}.$$

This describes N species of particles hopping on a lattice with hopping matrix element alternating between $t + \delta t$ and $t - \delta t$. This can be rewritten as

$$H = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \sum_{nm} \mathcal{H}_{nm}^{\alpha\beta} a_n^{\dagger\,(\alpha)} a_m^{(\beta)}, \tag{3.29}$$

with appropriately chosen $\mathcal{H}_{nm}^{\alpha\beta}$.

This system has a particular symmetry which protects its topological properties. It is called chiral symmetry, and is realized by the matrix

$$\Sigma_{nm} = (-1)^n \delta_{nm} \delta^{\alpha\beta}. \tag{3.30}$$

One can check that

$$\Sigma \mathcal{H} \Sigma = -\mathcal{H}. \tag{3.31}$$

This symmetry, in combination with time reversal invariance of this Hamiltonian, puts it in the symmetry class BDI of topological insulators, in one dimensional space. Those are characterized by an integer topological invariant.

One can compute the topological invariant of this problem, by going into the momentum space and finding that $\mathcal{H}$ becomes

$$\mathcal{H} = \delta^{\alpha\beta} \begin{pmatrix} 0 & t - \delta t + e^{ik} (t + \delta t) \\ t - \delta t + e^{-ik} (t + \delta t) & 0 \end{pmatrix}. \tag{3.32}$$

where k is the quasimomentum (crystalline momentum). At the same time, Σ becomes

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.33}$$

The topological invariant then can be computed according to

$$I = \mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \mathcal{H}^{-1} \partial_k \mathcal{H} \right] = \frac{N}{2} \left(1 + \mathrm{sign}\, \delta t\right). \tag{3.34}$$

That is, it is equal to N if δt is positive and to 0 if it is negative. As δt changes sign the system undergoes a phase transition (the gap closes at exactly $\delta t = 0$ as can be found from finding the energy spectrum of $\mathcal{H}$) from one phase to another. All of this is well known in the theory of topological insulators and was discussed in other lectures at the school.

Note that the chiral symmetry is crucial in making the invariant I to take on integer values, and to change only when the gap closes. Indeed, if the matrix $\mathcal{H}$ is perturbed slightly according to $\mathcal{H} \to \mathcal{H} + \delta\mathcal{H}$, the change in I can be computed according to

$$\delta I = \mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \mathcal{H}^{-1} \partial_k \left(\delta \mathcal{H} \right) \right] + \mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \delta \mathcal{H}^{-1} \partial_k \mathcal{H} \right] = \tag{3.35}$$

$$\mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \mathcal{H}^{-1} \partial_k \left(\delta \mathcal{H} \right) \right] - \mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \mathcal{H}^{-1} \delta \mathcal{H} \mathcal{H}^{-1} \partial_k \mathcal{H} \right] =$$

$$\mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \mathcal{H}^{-1} \partial_k \mathcal{H} \mathcal{H}^{-1} \delta \mathcal{H} \right] - \mathrm{tr} \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \left[\Sigma \mathcal{H}^{-1} \delta \mathcal{H} \mathcal{H}^{-1} \partial_k \mathcal{H} \right] = 0.$$

Here we used that

$$\partial_k \mathcal{H}^{-1} = -\mathcal{H}^{-1} \partial_k \mathcal{H} \mathcal{H}^{-1}, \ \delta \mathcal{H}^{-1} = -\mathcal{H}^{-1} \delta \mathcal{H} \mathcal{H}^{-1}. \tag{3.36}$$

We also used the symmetry (3.31) to permute Σ and $\mathcal{H}$ and the cyclic permutation symmetry of the trace.

What follows from it is that I is insensitive to the details of $\mathcal{H}$ (topological invariant) and does not change if cH is changed. It can only change if $\mathcal{H}$ becomes singular, that is, develops zero eigenvalue and becomes noninvertible, which is equivalent to closing the gap in the spectrum.

If the Hamiltonian is perturbed so that the symmetry (3.31) no longer holds, I is no longer an integer valued invariant. By chaning $\mathcal{H}$ it will be possible to make I change its values without closing the gap. That's why this is an SPT phase, protected by symmetry (3.31).

3.3 Interacting SPT phases: SSH-Hubbard model

In this section we will follow the work Phys. Rev. B **86**, 205119 (2012).

While classification of distinct topological insulator phases is well known, and is based on classifying all possible topological invariants, the question remains what happens to this classification if interactions are added to the Hamiltonian. It turns out that often two phases which were distinct without interactions (could not be converted into each other without closing the gap) become the same phase once interactions are turned on (can now be converted into each other without closing the gap even if the symmetry protecting these phases is preserved). Let us illustrate this on the example of the interacting SSH model.

We consider the model Eq. (3.27) and add Hubbard-like on-site interactions to it.

$$H = \sum_{\alpha=1}^{N} \sum_{n} (t + (-1)^n \delta t)\, a_n^{\dagger\,(\alpha)} a_{n+1}^{(\alpha)} + \text{h.c.} + \frac{U}{2} \sum_{n} \sum_{\alpha\beta} a_n^{\dagger\,(\alpha)} a_n^{(\alpha)} a_n^{\dagger\,(\beta)} a_n^{(\beta)} - \mu \sum_{n} \sum_{\alpha} a_n^{\dagger\,(\alpha)} a_n^{(\alpha)}. \quad (3.37)$$

Note that the interaction terms is nothing but the square of the number of particles on a given site, which naturally happens when particles interacting via 2-body interactions are piled in one site.

This describes particles which obey SSH model but also interact when they occupy the same site, via Hubbard interactions among all species of particles. We also add a chemical potential μ term which controls how many particles we have in the system since we are now dealing with genuine interacting many body system.

Later we will consider somewhat more general interactions which have a more complicated structure in the space of α and β than above, but for now let us look at these specific interactions.

The chiral symmetry present in the noninteracting version of this system generalizes to the presence of interactions, if μ is chosen correctly. One can check that this model is invariant under

$$a_n^{(\alpha)} \to (-1)^n a_n^{\dagger\,(\alpha)}, \;\; a_n^{\dagger\,(\alpha)} \to (-1)^n a_n^{(\alpha)} \quad (3.38)$$

leaves the Hamiltonian invariant as long as μ is chosen as

$$\mu = \frac{U}{2}. \quad (3.39)$$

(Technically speaking, (3.38) is equivalent to (3.31) only for real Hamiltonians, that is, in the presence of time-reversal symmetry, but the experts will recognize in (3.38) the particle-hole symmetry, equivalent to chiral symmetry for real, or more generally, time-reversal invariant, Hamiltonians. Fortunately, (3.37) is real, so we will not make this distinction.)

Now it should also be clear that if a Hamiltonian is invariant under (3.38), it is invariant under replacing all particles by holes and vice versa, that is, it is exactly at half filling. One can verify that (3.39) is exactly the right value of μ to place us at exactly $N/2$ particles per lattice site on the average.

Now we will follow the strategy first described by Fidkowski and Kitaev. Let us consider an extreme case when $\delta t = t$. In that case, the very first, boundary, site of the lattice, completely decouples from the rest of the lattice, and its Hamiltonian becomes

$$H = \frac{U}{2} \sum_{\alpha\beta} a_1^{\dagger(\alpha)} a_1^{(\alpha)} a_1^{\dagger(\beta)} a_1^{(\beta)} - \frac{U}{2} \sum_{\alpha} a_1^{\dagger(\alpha)} a_1^{(\alpha)}. \tag{3.40}$$

This describes the boundary states of our problem (the index 1 is the boundary site $n = 1$). In the absence of interactions $U = 0$ the Hamiltonian above vanishes. The last site can be populated by any of the N species of particles. This means that the boundary state is highly degenerate, and the system has gapless excitations at the boundary (an excitation in this context would be replacing one particle by another, or adding a particle at the boundary site).

Suppose we now do have interactions, $U > 0$. Consider various values of N.

1. $N = 1$. There's only one particle, it cannot interact with itself and the problem has the same boundary states as the non-interacting problem. (Technically, this means that the first term in the Hamiltonian (3.40) can only take values 0 or 1, but the second one also takes values 0 or 1, in such a way that the two terms always cancel).

2. $N = 2$. We are at half filling. That means there's only one particle at the boundary site (half of 2). The same argument as above applies and the system has boundary states.

3. $N = 3$. There could be 3/2 of a particle at the boundary on the average, and again there could be no nontrivial interactions at the boundary.

4. $N = 4$. In this case, it is possible to arrange for clever interactions (going beyond (3.40) to completely remove zero energy excitations. We note that in this case, the boundary is occupied on the average by 2 particles. But altogether there could be 4 species of particles. Let us say that the first two particles is really the same particle, but with either spin-up or spin-down, and same for the second particle. But on the average only two particles

are present on the site, either the first one with either spin, or the second one with either spin. Instead of labeling these $a^{(\alpha)}$ with $\alpha = 1,2,3,4$, let's label them a_α with $\alpha =\uparrow,\downarrow$, and b_α with $\alpha =\uparrow,\downarrow$. Let us add to (3.40) the following spin-spin interactions, still on the same 1st site (so the site index is now suppressed to avoid clutter)

$$H_S = J \sum_{\alpha\beta} a^\dagger{}_\alpha \sigma^\mu_{\alpha\beta} a_\beta \; b^\dagger{}_\gamma \sigma^\mu_{\gamma\delta} b_\delta. \tag{3.41}$$

Here σ^μ, $\mu = 1,2,3$ are Pauli matrices. This is literally spin-spin interaction of the two particles, with $J > 0$. One can check that it is also invariant under (3.38).

We know very well that this interaction will have one ground state, and three excited states. Therefore, if the system is placed in the ground state, it will not have any gapless excitations in the boundary. This means that we thought it was a nontrivial topological state with $I = N$ (before we turned on interactions), but really it is equivalent to $I = 0$ state with no boundary gapless excitations.

One can verify actually that it is now possible to convert the $I = 0$ state into $I = 4$ state by first turning on J term, then changing the sign of δt, then switching off the J turn. Therefore, instead of the $\mathbb{Z}$ classification of the SPT phases without interactions, with interactions we only have 4 nontrivial SPT phases labeled by $\mathbb{Z}_4$. The phases with $I_2 - I_1 = 4n$ (n is arbitrary integer) which in the absence of interactions could not be converted into each other without closing the gap, now with interactions can be converted into each other without closing the gap.

We conclude that in the presence of interactions there are only 4 distinct one-dimensional phases of systems with chiral symmetry and time-reversal invariance (and with particle number conservation, as we will see below, that is with U(1) symmetry present in (3.37)). This should be contrasted with infinite number of phases labelled by N in the absence of interactions.

3.4 Classification of 1D SPT phases by projective representations of the symmetries

To draw more general conclusions on how to construct SPT phases protected by symmetries, let us look at a more general model whose only symmetry is time-reversal. We will construct it out of Majorana fermions (since it is easy in this language to break U(1) symmetry, that is, particle conservation).

Suppose we have a one dimensional lattice with Majorana fermions. We call them c_n and d_n. Those obey

$$c_n c_m + c_m c_n = \delta_{nm},\ d_n d_m + d_m d_n = \delta_{nm},\ c_n d_m + d_m c_n = 0. \tag{3.42}$$

Suppose now those obey the noninteracting Hamltonian

$$H = it_1 \sum_{n=1}^{L} c_n d_n + it_2 \sum_{n=1}^{L-1} d_n c_{n+1}. \tag{3.43}$$

i in front of terms is needed to make it Hermitian.

We note that it is possible to form the conventional fermionic creation and annihilation operators by following

$$a_n^\dagger = \frac{1}{2}\left(c_n + i d_n\right),\ a_n = \frac{1}{2}\left(c_n - i d_n\right), \tag{3.44}$$

as well known. Those obey the usual fermionic commutation relations as a consequence of (3.42).

Now time reversal operation leaves a_n and $a_n^\dagger$ invariant. At the same time, time-reversal is known to complex conjugate everything. It follows from here that d_n must change sign under time-reversal.

Let us now generalize the Hamiltonian to N species of Majoranas, with the result

$$H = it_1 \sum_{\alpha=1}^{N} \sum_{n=1}^{L} c_n^{(\alpha)} d_n^{(\alpha)} + it_2 \sum_{\alpha=1}^{N} \sum_{n=1}^{L-1} d_n^{(\alpha)} c_{n+1}^{(\alpha)}. \tag{3.45}$$

Clearly this is invariant under time reversal (it flips i in front of the Hamiltonian, but also flips the sign of all d). It also has a hidden particle-hole transformation because when written in terms of a_n, $a_n^\dagger$, it is a BdG superconductor with the particle non-conserving terms such as a^2. Their combination is the chiral symmetry, so it should be in the same class as the SSH model. And indeed, going to the extreme limit $t_1 = 0$ we see that the boundary Majoranas decouple from the Hamiltonian above (3.45). Creation and annihilation operators may be formed as

$$a^{(\alpha)} = \frac{1}{2}\left(c_1^{(\alpha)} + i d_L^{(\alpha)}\right). \tag{3.46}$$

Those annihilate (and their hermitian conjugates create) N fermions residing at the boundary forming zero energy excitations since their creation/annihilation does not cost any energy.

Note that is is impossible to add boundary quadratic terms to the Hamiltonian which would gap out those boundary excitations. Those would go as $ic_1^\alpha c_1^\beta$ (again, i due to the terms being hermitian), and those break time reversal so cannot be added as long as time-reversal is preserved. We recover the $\mathbb{Z}$ classification of the time-reversal non-interacting fermions with particle-hole or chiral symmetry.

If however we allow for terms with interaction which are at least quartic in Majoranas, we may expect that perhaps if there are 4 species of Majoranas the boundary can be gapped out. However, it is not so. We could construct the two fermionic operators at the boundary,

$$a_1^\dagger = \frac{1}{2}\left(c_1^{(1)} + ic_1^{(2)}\right),\ a_2^\dagger = \frac{1}{2}\left(c_1^{(3)} + ic_1^{(4)}\right). \tag{3.47}$$

We then note that we can now construct two states

$$|0\rangle\,,\ a_1^\dagger a_2^\dagger\,|0\rangle\,, \tag{3.48}$$

which form Kramers degenerate pair. We can see that by noting that time reversal conjugates i in (**??**) but does not touch the sign of c. Therefore, it exchanges a_1 and $a_1^\dagger$. As a result,

$$T\,|0\rangle = a_1^\dagger a_2^\dagger\,|0\rangle\,. \tag{3.49}$$

Here T is the time-reversal operation. Yet

$$T^2\,|0\rangle = a_1 a_2 a_1^\dagger a_2^\dagger\,|0\rangle = -\,|0\rangle\,. \tag{3.50}$$

No time reversal invariant term in the Hamiltonian can gap out some linear combination of these two states. It's because then the ground state is the other linear combination, and it must be time-reversal invariant, $T\,|GS\rangle = |GS\rangle$. This is not compatible with $T^2 = -1$ however. This is the essence of Kramers theorem.

This guarantees that there is going to be at least 2 ground states at the boundary which are degenerate. If one is the ground state, the other is a gapless excitation. So the $N = 4$ case is still topologically nontrivial with just as without interactions.

Note that these two states (3.48) can be thought of as spin-up and spin-down states which form a spin-1/2 multiplet. We can also think of them as the projective representation of the group responsible for time-reversal symmetry. Projective representation is the representation up to a phase. T repeated twice is identity, yet the spin-1/2 states get multiplied by -1 under the action of T^2, thus they form projective representation of time-reversal symmetry.

Only when $N = 8$, we can form a singlet (by combining two $N = 4$ spin-1/2s into a spin singlet) which is invariant under T. This can be chosen to be the ground state, while the rest of the states gapped out. At this stage, the $N = 8$ can be connected to $N = 0$ state. Therefore, we find that there are 8 distinct SPT phases in this system, for the total classification being $\mathbb{Z}_8$. This is the original result from Phys. Rev. B **81**, 134509 (2010).

The difference between $\mathbb{Z}_8$ here and $\mathbb{Z}_4$ in the previous subsection is the absence of U(1) conservation in this subsection.

That result is now generalized to claim that in one dimension the number of distinct SPT phases protected by a certain symmetry is related to the number of projective representations of this symmetry group.

This was conjectured to generalize to higher dimensions to give classification according to group cohomologies of symmetry groups. It turns out that this program gives partial but apparently not complete classification of higher dimensional SPT phases.

4 Floquet SPT phases

Some of these constructions were generalizes to systems with time dependent periodic Hamiltonians. Given $H(t)$ such that

$$H(t + T) = H(t), \tag{4.51}$$

we can construct an evolution operator over one period $U(T)$. It's a unitary operator, and its eigenvalues have the form $\lambda = e^{-i\epsilon T}$. If H is time independent, then ϵ are its eigenvalues, but more generally they are just the way to represent the eigenvalues of U. ϵ are usually called quasienergy because

$$\epsilon = \epsilon + \frac{2\pi}{T}. \tag{4.52}$$

Specializing to the 1D chiral systems, we note that the symmetry

$$\Sigma H \Sigma = -H \tag{4.53}$$

generalizes to

$$\Sigma U \Sigma = U^{-1}. \tag{4.54}$$

Because of that, in addition to the zero energy boundary states of 1D Floquet chiral systems, there could also be π/T energy boundary states (for which $\lambda^{-1} = \lambda$).

Because of that, the classification of the Floquet 1D chiral systems amounts to counting zero energy and "π-energy" states at the boundary, for the total classification of $\mathbb{Z} \times \mathbb{Z}$.

It can be shown however that in the presence of interactions (and in the absence of particle number conservation like in the example in the previous subsection) this reduces to

$$\mathbb{Z}_8 \times \mathbb{Z}_4. \tag{4.55}$$

For more detail, please see Phys. Rev. X **6**, 041001 (2016).