New Zealand INSTITUTE for Advanced Study

CTCP
Centre for Theoretical Chemistry and Physics

Exotic topological states of ultra-cold atomic matter
Lecture 1: Topolgical and non-topological solitons

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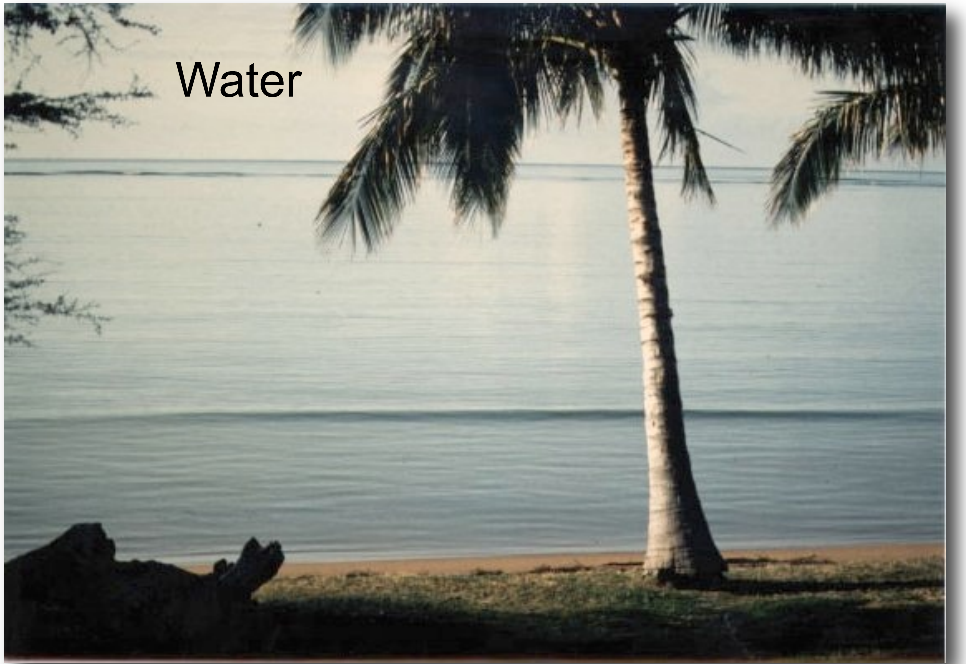
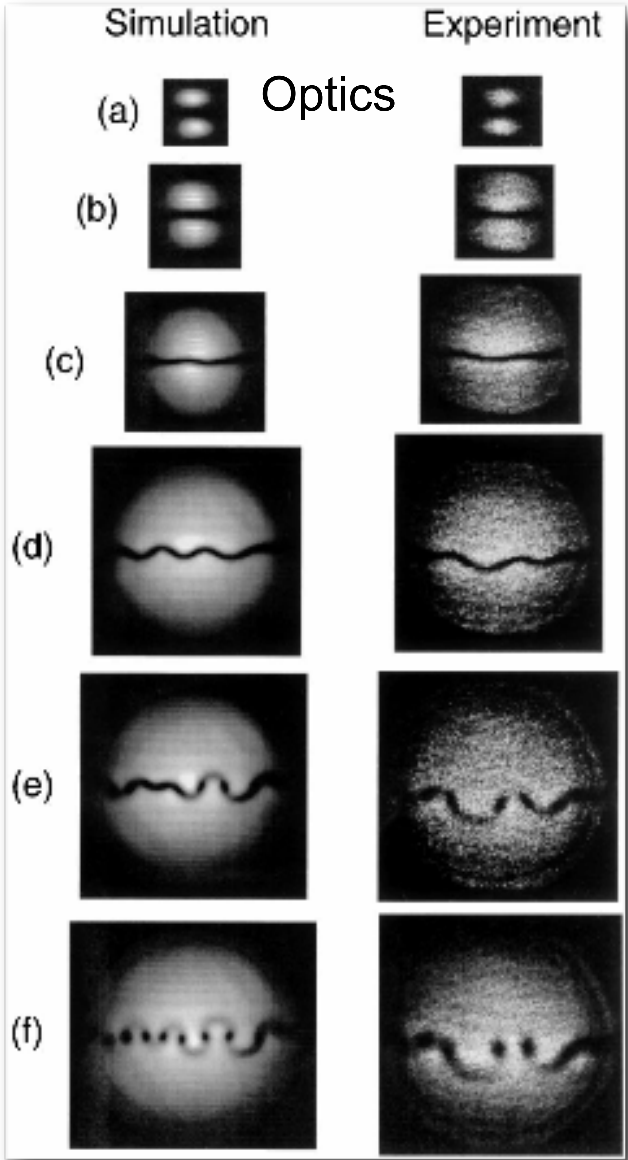
COLLEGE OF SCIENCES

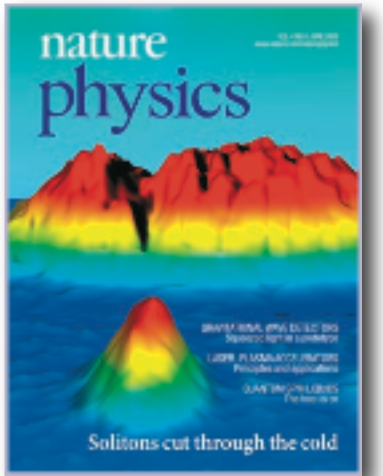
To be covered: Solitons in quantum gases

- **Lecture 1:** Solitons and topological solitons
  - solitons in water: the KdV equation, iintegrability
  - solitons of the nonlinear Schrodinger equation
  - solitons of the sine Gordon equation - topological solitons
  - Bose Josephson vortices in linearly coupled BECs
- **Lecture 2:** Semitopological solitons in multiple dimension
  - Solitons as quasiparticles: effective mass
  - solitons in the strongly-interacting Fermi gas
  - snaking instability
  - vortex rings
  - solitonic vortices
- **Lecture 3:** Quantum solitons and Majorana solitons
  - solitons in strongly-correlated 1D quantum gas
  - solitons with Majorana quasiparticles in fermionic superfluids

Solitons





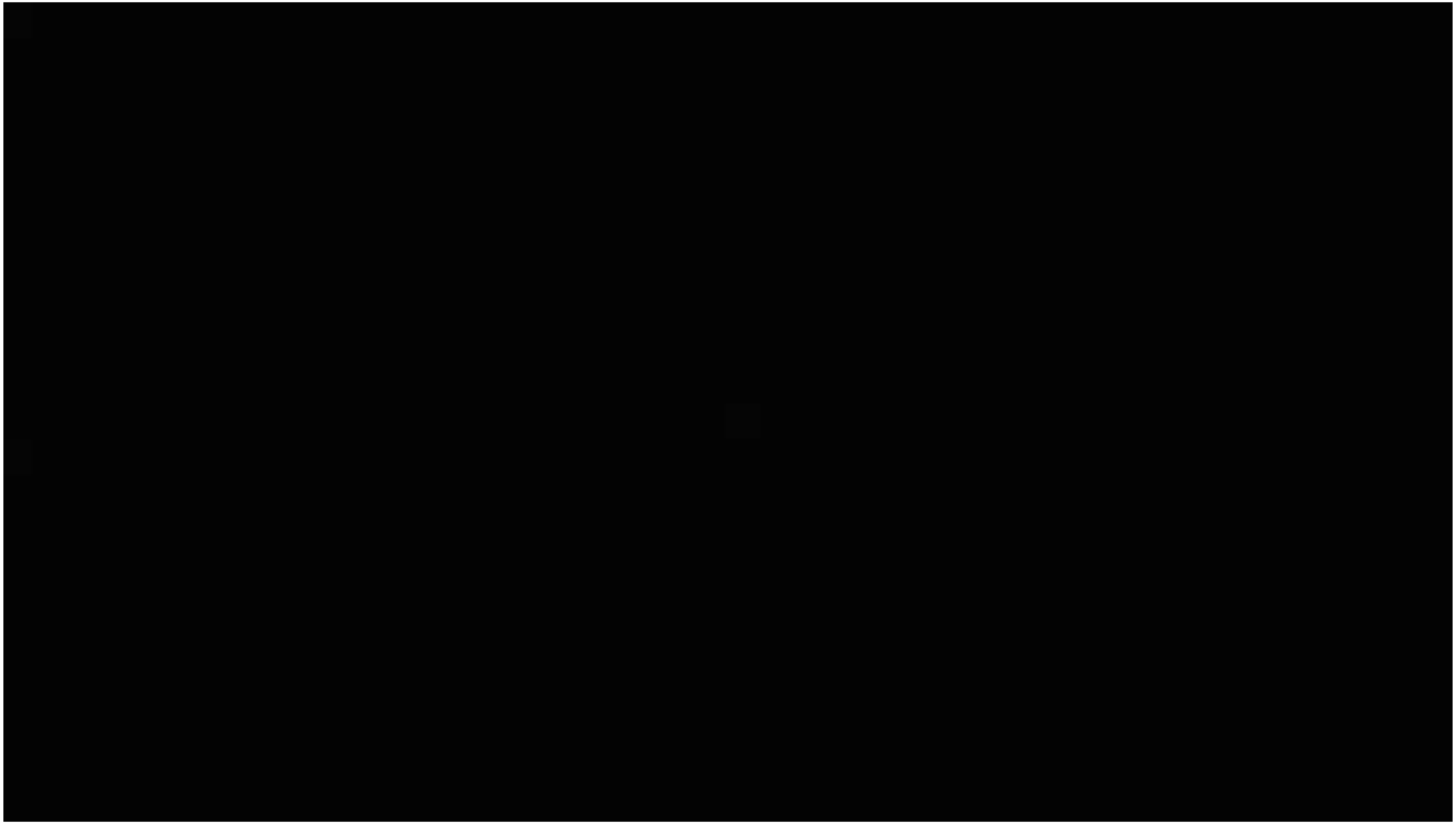


BEC

Topologial solitons

So if a soliton is a localised wave, then what is a topological soliton?

Hagfish makes a knot



Credit: Stefan Siebert, Sophia Tintory, Casey Dunn   https://vimeo.com/7825337

Topologial solitons

So if a soliton is a localised wave, then what is a topological soliton?

Wikipedia:
"A **topological soliton** or a **topological defect** is a solution of a system of partial differential equations or of a quantum field theory homotopically distinct from the vacuum solution."

Homotopy: a continuous deformation

Solitons appear spontaneously

e.g. when cooling through the Bose-Einstein condensation phase transition

- Nature Physics 2013:

nature physics

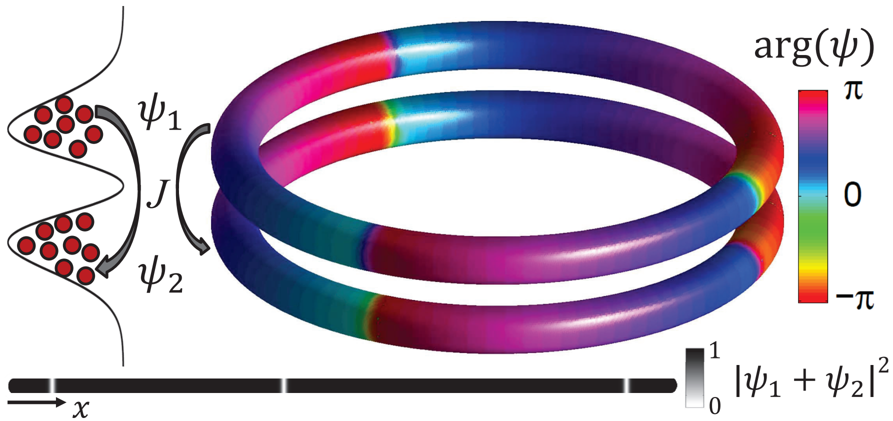
ARTICLES

PUBLISHED ONLINE: 8 SEPTEMBER 2013 | DOI: 10.1038/NPHYS2734

Spontaneous creation of Kibble–Zurek solitons in a Bose–Einstein condensate

Giacomo Lamporesi, Simone Donadello, Simone Serafini, Franco Dalfovo and Gabriele Ferrari*

Also: proposal to observe Josephson vortices (topological solitons) by rapidly cooling a double-ring Bose-Einstein condensate.



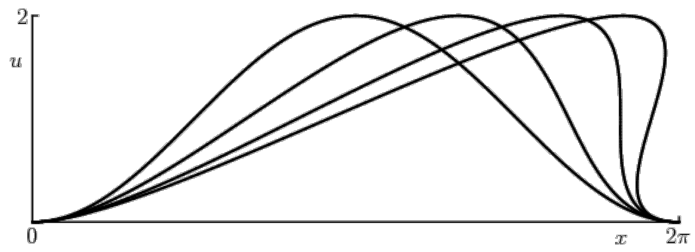
SW Su, SC Gou, AS Bradley, O Fialko, JB, Phys. Rev. Lett. **110**, 215302 (2013)

From linear to nonlinear waves: shallow water

$\partial_t \phi + c\,\partial_x \phi = 0$ Linear wave equation $\phi(x,t) = A\sin(x - ct)$

Nonlinear waves: wave speed depends on amplitude:

$\partial_t \phi + \phi\,\partial_x \phi = 0$ Inviscid Burgers equation



Source: Leon van Dommelen, FSU

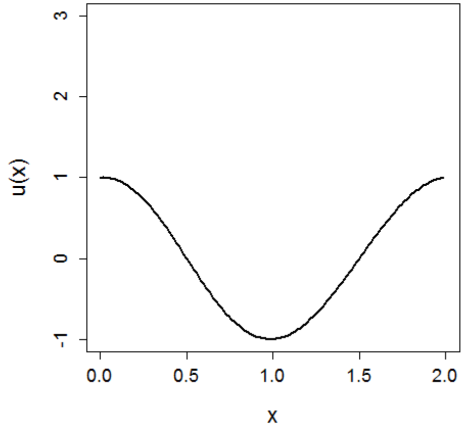
Add dispersive (higher order derivative term):

Korteweg – de Vries equation (1895)

$$\partial_t \phi + \partial_x^3 \phi + 6\phi\,\partial_x \phi = 0$$

Soliton solution

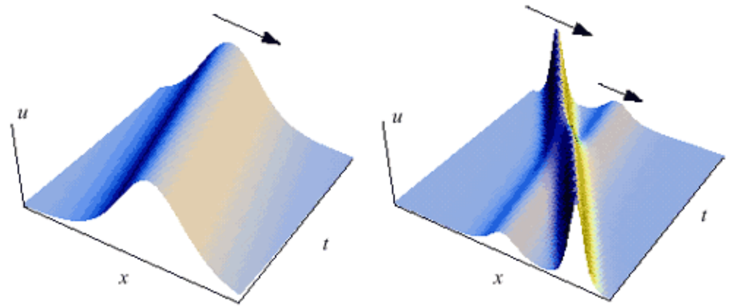
$$\phi(x,t) = \frac{1}{2} c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct - a)\right]$$



Source: Wikipedia

KdV: an integrable soliton equation

1965: Zabusky and Kruskal discover
robust collision in numerics,
invent the term "soliton"



KdV 1-soliton KdV 2-soliton

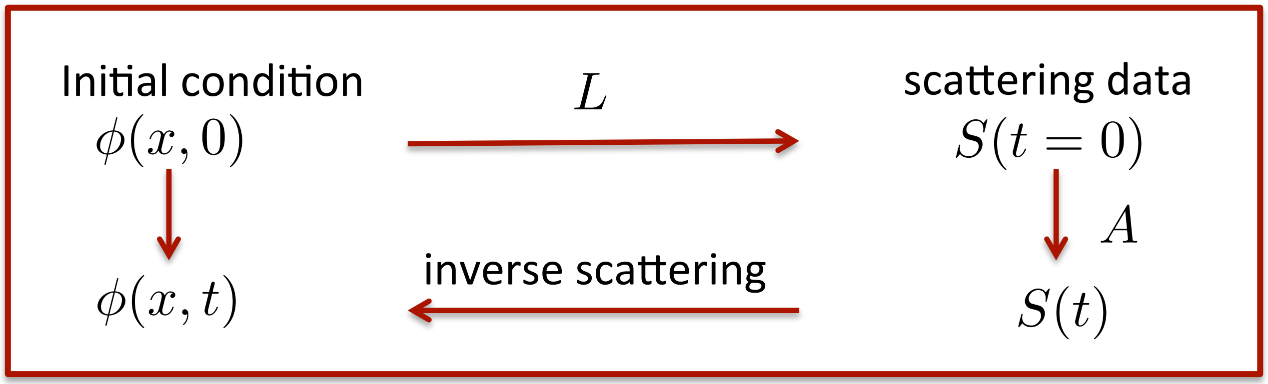
1967: Inverse scattering transform (Gardner, Greene, Kruskal, Miura)
is based on the existence of a

Lax pair

$$L = -\partial_x^2 + \phi$$

$$A = 4\partial_x^3 - 3[2\phi\partial_x + (\partial_x\phi)]$$

$$\partial_t L = [L, A]$$



Inverse scattering transform method

The scattering problem

The linear Schrödinger equation

$$L\psi(x) = \lambda\psi(x) \qquad \text{with} \qquad L = -\partial_x^2 + \phi$$

has bound state solutions $\lambda_i < 0$ "solitons"

and scattering states $\lambda \geq 0$ "radiation"

The nature of the scattering problem does not change as time evolves, thus solitons are eternal. Moreover, there is an infinite number of constants of the motion – the problem is *integrable*.

Long term fate of a localised initial state (finite support) $\phi(x,0)$

For $\phi(x,t)$ with $t \to \infty$

- Solitons will persist, separate
- Radiation will decay to zero amplitude

Examples of integrable soliton equations

- **Korteweg – de Vries equation**:

$$\partial_t \phi + \partial_x^3 \phi + 6\phi\, \partial_x \phi = 0$$

real wave function, bright solitons only

- **Nonlinear Schrödinger equation:**

$$i\partial_t u = -\partial_x^2 u \pm |u|^2 u$$

complex wave function, bright and dark solitons

- **Sine Gordon equation:**

$$\partial_t^2 \phi - \partial_x^2 \phi + \sin(\phi) = 0$$

relativistic covariant wave equation (Lorentz transformation);
real wave function, topological solitons

Theory: Bose-Einstein Condensate (BEC)

- Bose gas in an external potential

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\boldsymbol{r},t)=\left[-\frac{\hbar^2}{2m}\nabla^2+V_{\text{ext}}(\boldsymbol{r},t)+\int\hat{\Psi}^\dagger(\boldsymbol{r}',t)V(\boldsymbol{r}'-\boldsymbol{r})\hat{\Psi}(\boldsymbol{r}',t)\mathrm{d}\boldsymbol{r}'\right]\hat{\Psi}(\boldsymbol{r},t)$$

For BECs we may use the classical or mean field (Hartree) approximation:

Interaction becomes a tunable parameter

Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r},t)=\left[-\frac{\hbar^2}{2m}\nabla^2+V_{\text{ext}}(\boldsymbol{r},t)+\frac{4\pi a_s}{m}|\psi(\boldsymbol{r},t)|^2\right]\psi(\boldsymbol{r},t)$$

a_s s-wave scattering length

The GP equation is a *nonlinear Schrödinger equation*

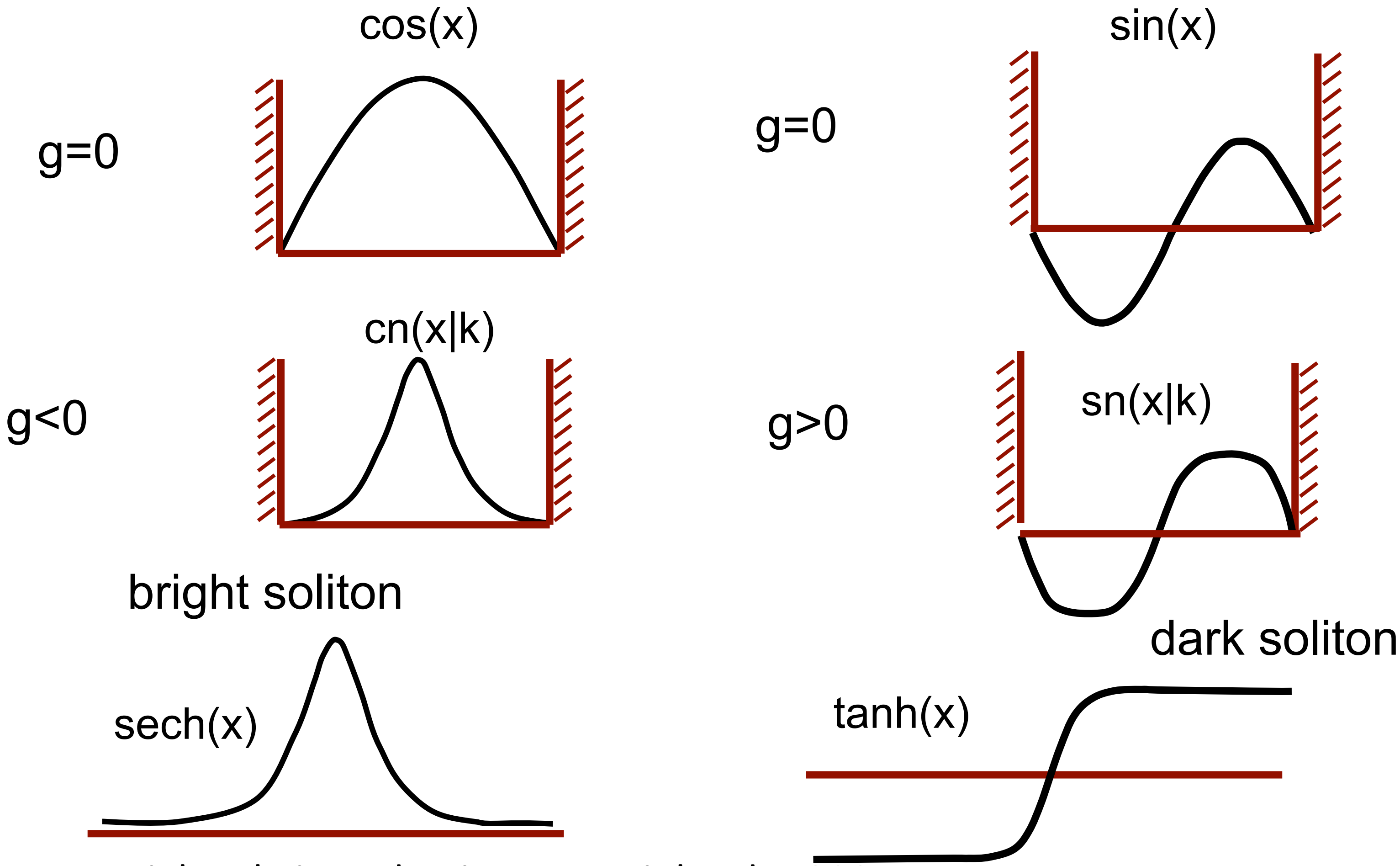
Is GP valid for soliton phenomena?

Criterium of validity:

healing length
length scale for solitons $\xi=\frac{1}{\sqrt{8\pi n|a_s|}}\gg d$ particle distance

Solitons as stationary solutions of the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}u(x,t) = [-\frac{1}{2}\frac{\partial^2}{\partial x^2} + g|u|^2]u(x,t)$$



For a tutorial-style introduction see Reinhardt 1988

Solitons

in the nonlinear Schrödinger equation (NLS)

Dispersion

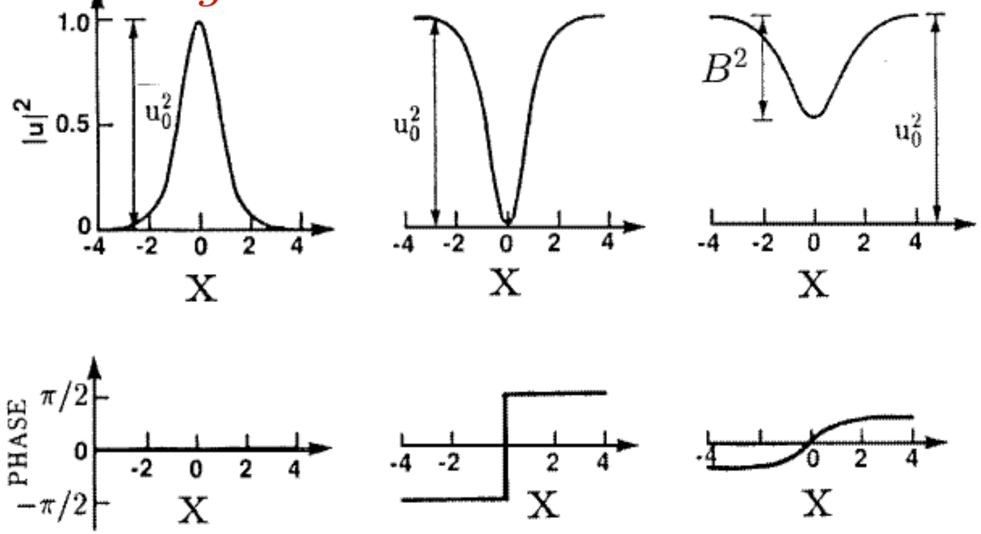
$$i\frac{\partial}{\partial t}u(x,t) = [-\frac{1}{2}\frac{\partial^2}{\partial x^2} + g|u|^2]u(x,t)$$

Nonlinearity

bright soliton $g < 0$

dark solitons $g > 0$

$$u(x,t) = u_0\{Ai + B\tanh[u_0 B(x - Au_0 t)]\}e^{iu_0^2 t}$$



$$A^2 + B^2 = 1$$

Phase step

$$\Delta\phi = 2\tan^{-1}\left(\frac{A}{B}\right)$$

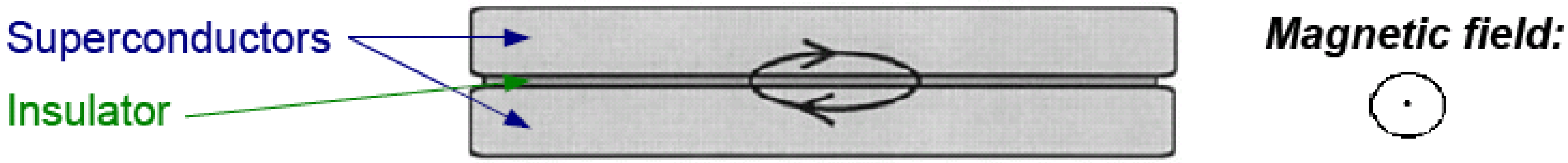
From: Kivshar (1998)

Solitons in quantum gases

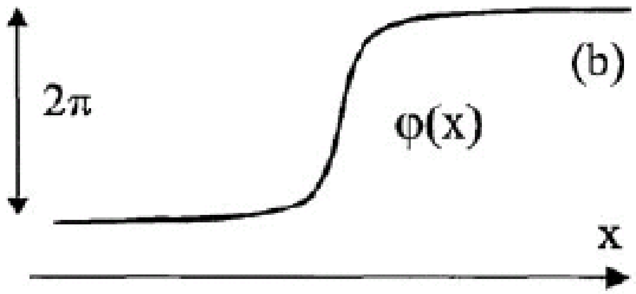
- Bose-Einstein condensate in quasi-1D trap: Gross-Pitaevskii equation -> NLS
  - Dark solitons with repulsive interactions
  - Bright solitons with attractive interactions
- Superfluid Fermi gas in BEC – BCS crossover
  - BEC regime -> dark solitons as above (NLS) in quasi 1D
  - BCS regime -> Bogoliubov-de Gennes equation with dark soliton solutions in 1D
  - Unitary regime, 3D, strictly 1D -> to be discussed
- Linearly coupled 1D BECs -> coupled 1D GPEs
  - Not integrable but features both NLS and sine Gordon solitons

Josephson vortices in superconductor

Long Josephson juction



- ***Josephson vortex:*** identified by a soliton in the relative phase



- One quantum of magnetic flux

A. V. Ustinov, Physica D, ***123*** *(1998)*

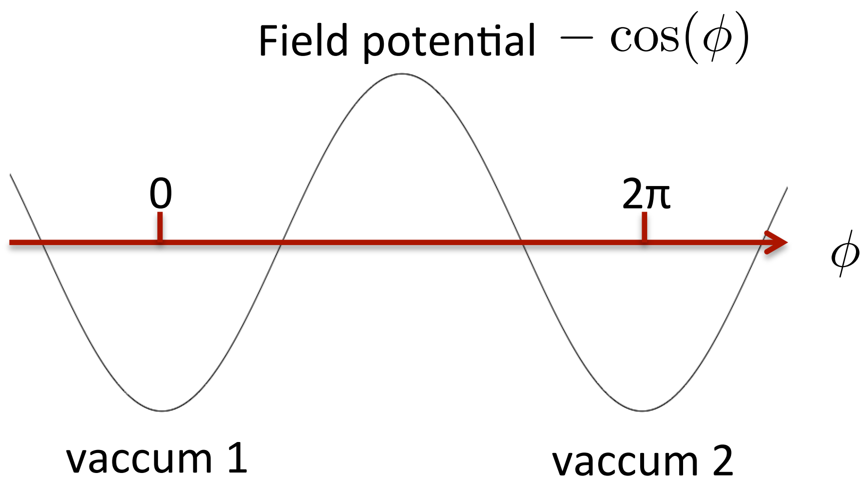
Solitons of the sine Gordon equation

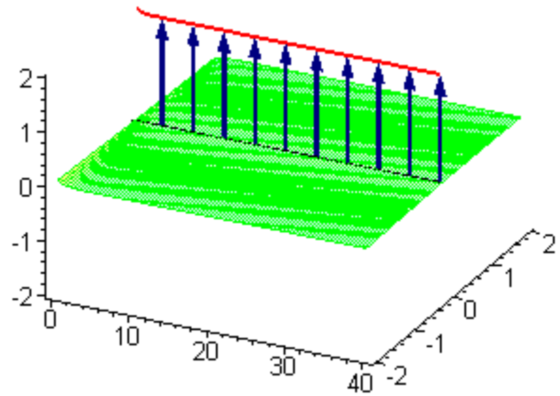
The sine Gordon equation

$$\partial_t^2\phi - \partial_x^2\phi + \sin(\phi) = 0$$

corresponds to the energy density

$$w = \frac{1}{2}(\partial_x\phi)^2 - \cos(\phi)$$





The sine Gordon kink is a topological soliton.
It connects two vacuua.

Classification of solitons

- **Non-topological soliton:**
  relies on the balance of nonlinearity and dispersion

- **Topological soliton:**
  owes its existence to a multiplicity of ground states that allow topologically non-trivial field configurations

Topological charge for sine-Gordon:

$$Q = \frac{1}{2\pi}\left[\phi(x=\infty) - \phi(x=-\infty)\right]$$

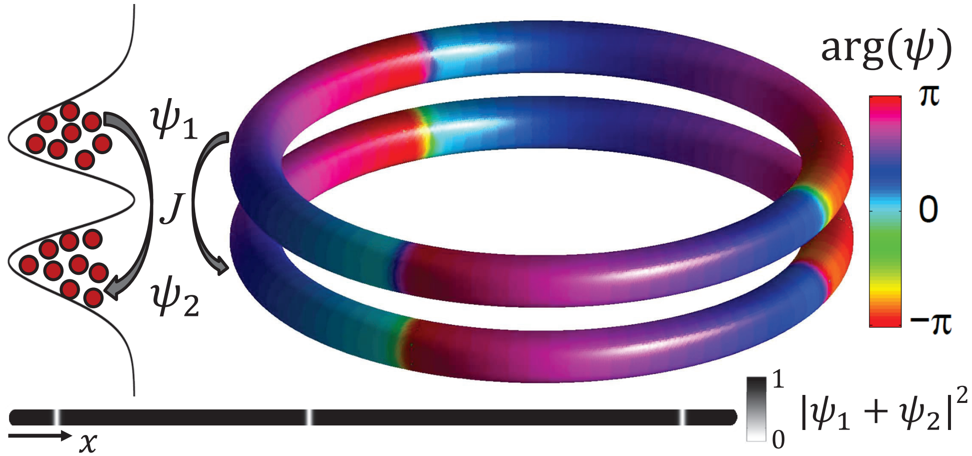
Associated conserved current:

$$j = \frac{1}{2\pi}\frac{\partial \phi}{\partial x} \qquad\qquad Q = \int_{-\infty}^{\infty} j\, dx$$

Two coupled Bose fields

$$i\hbar\partial_t\psi_1 = -\frac{\hbar^2}{2m}\partial_{xx}\psi_1 - \mu\psi_1 + g|\psi_1|^2\psi_1 - J\psi_2$$
$$i\hbar\partial_t\psi_2 = -\frac{\hbar^2}{2m}\partial_{xx}\psi_2 - \mu\psi_2 + g|\psi_2|^2\psi_2 - J\psi_1$$

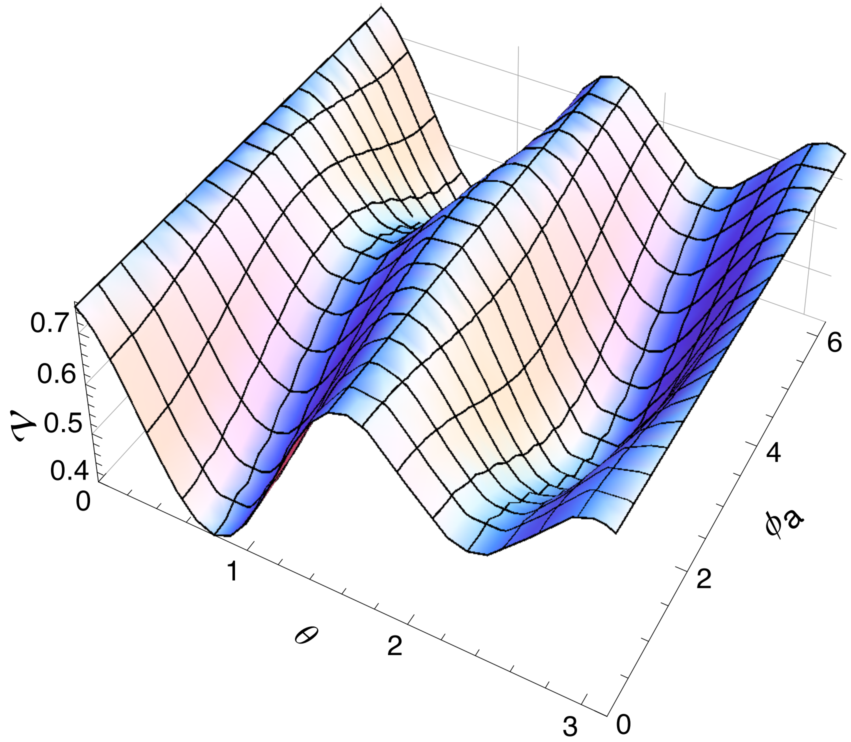
J is tunnel coupling
μ is the chemical potential
g>0 interaction between atoms



Important parameter: $\nu = \frac{J}{\mu}$

Could be realised in double ring trap or two linear traps with narrow barrier (Schmiedmayer experiments).

Field potential for coupled BECs

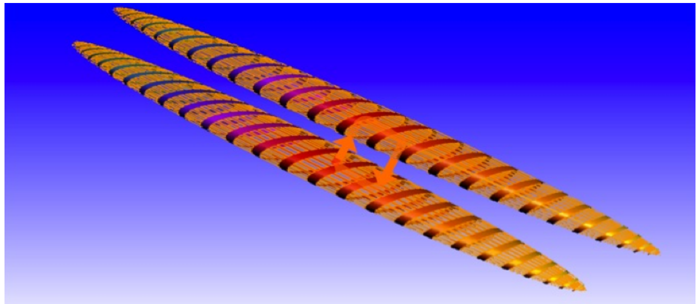


Field potential for coupled BEC fields

- Relative phase and amplitude yield sine-Gordon equation – a relativistic field theory!
- Total phase and density yield nonlinear Schrödinger equation – with dark solitons and phonons.

B Opanchuk, R Polkinghorne, O Fialko, JB, P Drummond, Ann Phys. (Berlin) (2013)

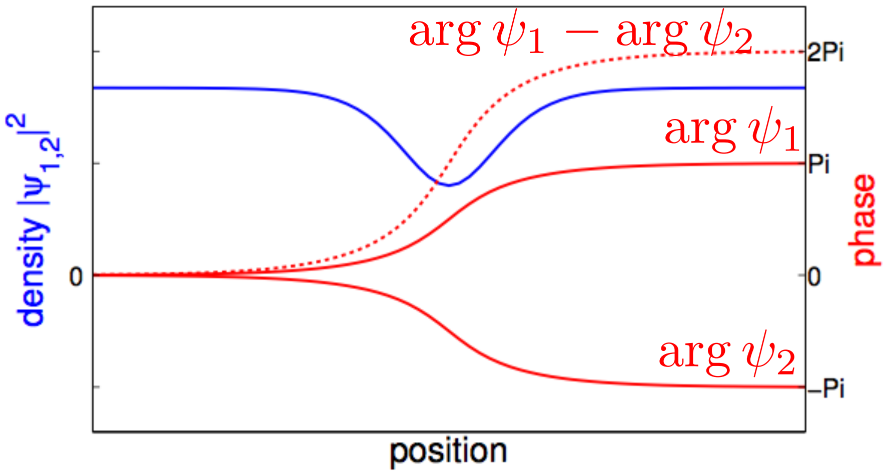
Josephson vortex and dark soliton



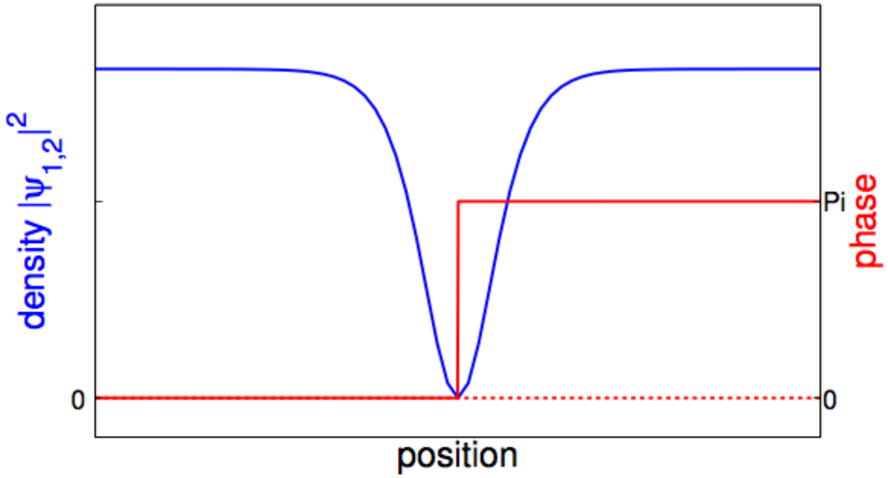
$$i\hbar\partial_t\psi_1 = -\frac{\hbar^2}{2m}\partial_{xx}\psi_1 - \mu\psi_1 + g|\psi_1|^2\psi_1 - J\psi_2$$

$$i\hbar\partial_t\psi_2 = -\frac{\hbar^2}{2m}\partial_{xx}\psi_2 - \mu\psi_2 + g|\psi_2|^2\psi_2 - J\psi_1$$

Josephson vortex



Dark soliton

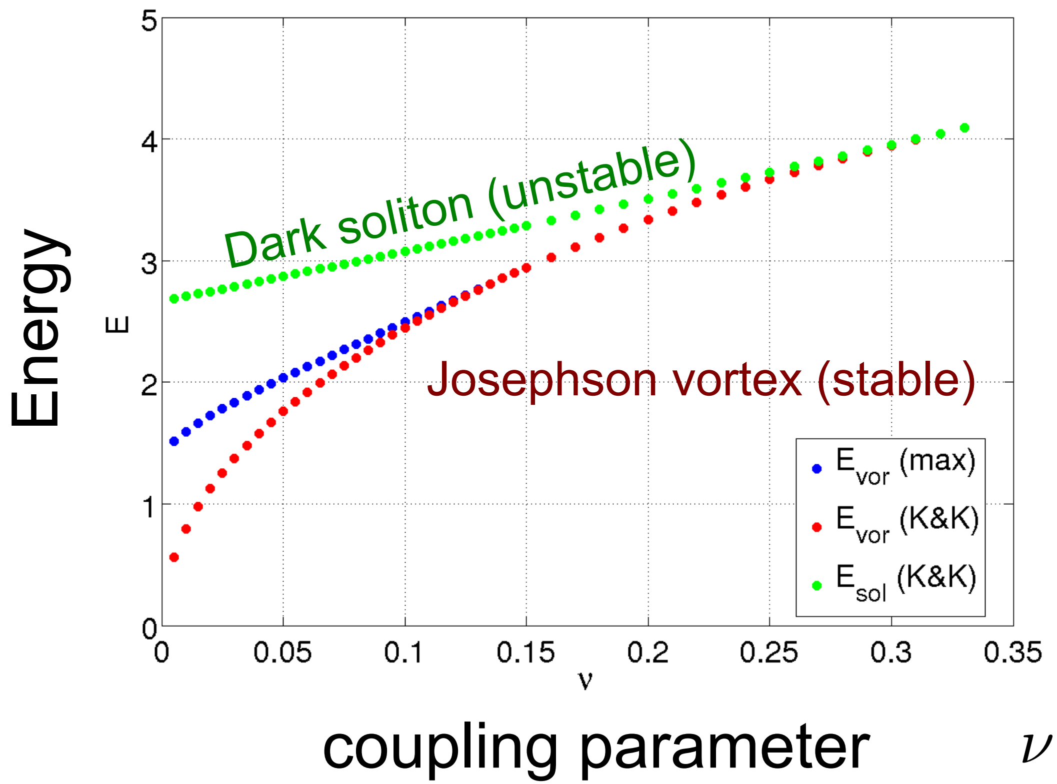


The stationary solutions were found by Kaurov and Kuklov PRA (2005)

Related: JB,T Haigh, U Zuelicke PRA 2009

L Wen, H Xiong, B Wu PRA 2010

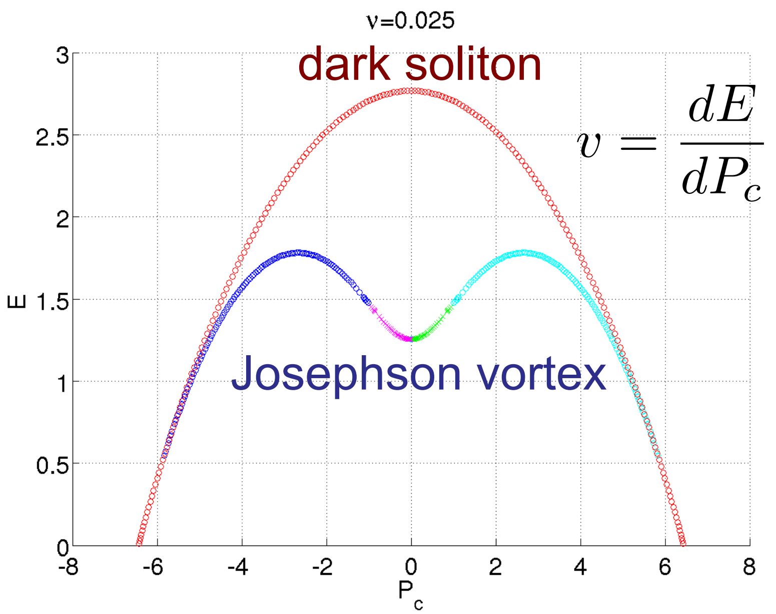
Josephson vortex vs dark soliton

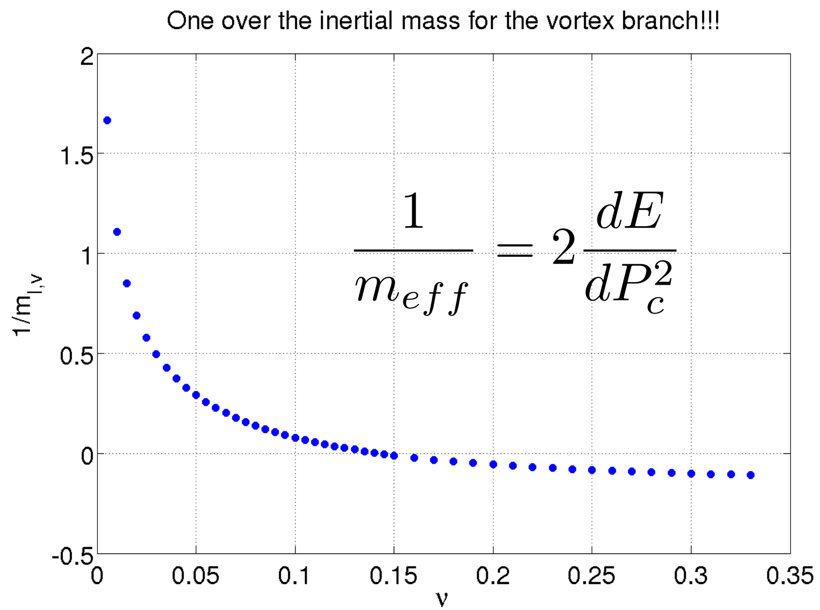


$$\nu = \frac{J}{\mu}$$

Josephson vortex dispersion

Josephson vortices can move





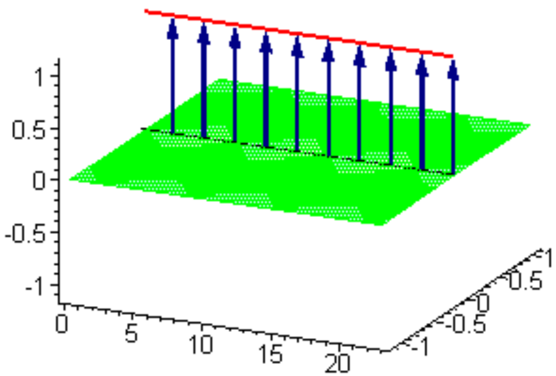
They are quasiparticles with tunable effective mass

Sophie Shamailov and JB, arXiv:1709.00403

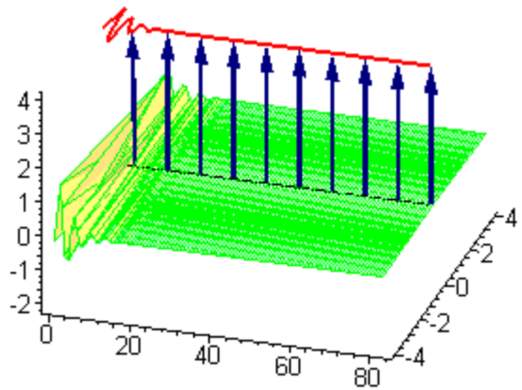
Breathers and oscillons

- **Breathers** in the sine Gordon equation are not topological, but live forever

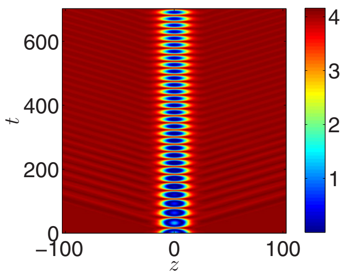
Stationary large-amplitude breather

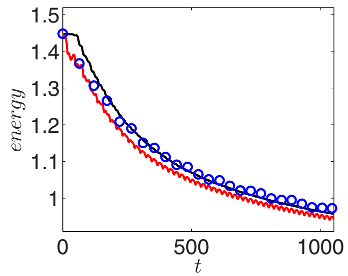


Small-amplitude breather



- In the coupled BECs, instead we find **oscillons**: breather-like excitations that live a long time





S-W Su, S-C Gou, I-K Liu, AS Bradley, O Fialko, JB, PRA (2015)

Examples of integrable soliton equations

- **Korteweg – de Vries equation**: water waves

  $$\partial_t \phi + \partial_x^3 \phi + 6\phi\, \partial_x \phi = 0$$

- **Focusing nonlinear Schrödinger equation:**

  $$i\partial_t u = -\partial_x^2 u - |u|^2 u$$

  Attractive Bose-Einstein condensates in quasi-1D waveguide
  Experiments by Hulet, Salomon, Cornish, Kasevich

- **Defocusing nonlinear Schrödinger equation:**

  $$i\partial_t u = -\partial_x^2 u + |u|^2 u$$

  Repulsively interacting Bose-Einstein condensates
  Experiments by Sengstock, Phillips, Oberthaler

- **Sine Gordon equation:**

  $$\partial_t^2 \phi - \partial_x^2 \phi + \sin(\phi) = 0$$

  Realised by linearly coupled Bose-Einstein condensates (Schmiedmayer experiments?)