DRAFT: Thursday 15th December, 2011, 11:21

Introduction to Gauge/Gravity Duality

Lecture 3

Nick Halmagyi

Center for the Fundamental Laws of Nature
Harvard University
Cambridge, MA, 02138 USA

Laboratoire de Physique Théorique et Hautes Energies,
Université Pierre et Marie Curie, CNRS UMR 7589,
F–75252 Paris Cedex 05, France

Abstract

A series of four lectures given at the Australian National University in December 2011 for the 2nd Asia-Pacific Summer School in Mathematical Physics/22nd Canberra International Physics Summer School.

1 The Conformal Group

It is quite possible that you are familiar with conformal symmetry from the study of two dimensional quantum field theory, where conformal symmetry places stringent constraints on correlation functions. At risk of duplication, we will review conformal symmetry here since it plays a crucial role in holography.

Perhaps the Poincaré group is familiar from undergraduate study of relativistic quantum mechanics. This group is generated by Lorentz transformations $M_{\mu\nu}$ and translations P_μ and technically we say that this is

$$O(1,3) \ltimes \mathbb{R}^{1,3} \tag{1}$$

to indicate that $O(1,3)$ is a normal subgroup. This is the *minimal* symmetry group of a relativistic classical field theory but there is a large swath of modern physics which deals with field theories which possess a larger symmetry group.

A canonical symmetry generator to consider is dilatation:

$$D : x^\mu \to \lambda x^\mu \,, \quad \lambda \in \mathbb{R}_+ \,. \tag{2}$$

For such a symmetry to be present in a given quantum field theory, the spectrum of fields must be massless since there cannot be any dimensionful parameters. A dimensionful parameter would set a prefered scale and thus by definition break scale invariance.

A common technique to enhance the symmetry of a given system is the make a *global* symmetry *local*, this is often refered to as *gauging* a symmetry. To see this more precisely we will just present a defintion of the conformal group. It is the group of diffeomorphisms which leave the metric invariant up to an overall local rescaling:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \omega^2(x) g_{\mu\nu}(x) \,. \tag{3}$$

In Minkowski space, it should be clear that the Poincaré group leaves the metric exactly invariant. The full conformal group has one more generator

$$K : x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a_\mu x^\mu + a^2 x^2} \tag{4}$$

which goes by the rather unimaginitive moniker *special conformal transformation*. This is a fairly unituitive transformation but one can check that

$$K : \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + a^\mu \,. \tag{5}$$

For dilations and special conformal transformation we find that

$$\begin{aligned} D: \ \omega(x) &= \lambda^{-1} \,, && (6)\\ K_\mu: \ \omega(x) &= (1 + 2a_\mu x^\mu + a^2x^2)^{-1} \,. && (7) \end{aligned}$$

For reference we include here the full algebra of the conformal group:

$$\begin{aligned} \big[P_\rho, M_{\mu\nu}\big] &= i\big(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\big) && (8)\\ \big[K_\rho, M_{\mu\nu}\big] &= i\big(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\big) && (9)\\ \big[M_{\mu\nu}, M_{\rho\sigma}\big] &= i\big(\eta_{\nu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\nu\rho} + \eta_{\nu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\nu\rho}\big) && (10)\\ \big[D, M_{\mu\nu}\big] &= 0 && (11)\\ \big[D, K_\mu\big] &= iK_\mu && (12)\\ \big[D, P_\mu\big] &= -iP_\mu && (13)\\ \big[P_\mu, K_\nu\big] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu} D && (14) \end{aligned}$$

2 The Conformal Fields of Conformal Field Theory

Having learnt previously about how the conformal group acts on spacetime, it is necessary now to understand how the symmetry group acts on the fields of our QFT. The guiding principle when realizing symmetry groups on a space of physical fields is that the fields must furnish a *unitary* representation. When the symmetry group is the conformal group, these interesting representations are constructed from so-called *primary* operators $\Phi(x)$. These operators have two properties

- $\Phi(x)$ are annihilated by the generator of special conformal transformation K_μ
- $\Phi(x)$ are eigenfields of the dilation operator D with eigenvalue $-i\Delta$

To understand these representations one should recall the highest weight representations of $SU(2)$, familiar from quantum mechanics. In this case, one has three operators (J_z, J_+, J_-) and one calls a *spin* j representation one where a particle can have the following eigenvalues under J_z:

$$j, j-1, \ldots, 1-j, -j\,. \tag{15}$$

Then of course $J_\pm$ raise and lower the spin by one unit. One key point is that there are only finitely many eigenstates since the spectrum is bounded above and below, in particular it is symmetric around spin zero.

Coming back to the conformal group, there are some similiarities and some important differences. The primary fields have the lowest possible dimension in the representation and the by unitarity the spectrum is bounded below by

$$\Delta \geq (d-2)/2. \tag{16}$$

It is not bounded above however, which is perfectly fine, QFT does not have a finite spectrum. If you are familiar with the state-operator correspondence as well as the infinite dimensional nature of the Hilbert space of states, this should not be a surprise. From (14) one might be able to discern that if the Lorentz transformation acts trivially (as it does for a realization on a scalar field) the this commutation relation is of the form

$$[J_+, J_-] = 2iJ_z \tag{17}$$

and indeed P_μ and K_μ should be thought of as raising and lowering operators and indeed the entire representation can be generated by acting with P_μ on a primary field. A very important point is that while for $SU(2)$ the allowed spins are quantized and integral, which is a result of the reflection symmetry about spin zero, the conformal dimension is neither quantized or integral. In fact it can depend on the coupling constants in the theory.

So lets work out how symmetries act on scalar fields. As usual for a scalar field we demand that it is invariant under Poincaré transformations (while of course spinors, vectors and tensors transform nontrivially) but it need not and in fact *should not*, be invariant under dilatations. We demand that for a field $\Phi(x)$

$$\Phi'(x') \stackrel{!}{=} \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \Phi(x) \tag{18}$$

$$= \omega(x)^\Delta \Phi(x) \tag{19}$$

where

$$\left|\frac{\partial x'}{\partial x}\right| = \omega(x)^{-d} \tag{20}$$

is the Jacobian for a given conformal transformation.

To first order in some set of parameters ϵ^a we get

$$\Phi'(x') = \Phi(x) + \epsilon^a \frac{\delta x^\mu}{\delta \epsilon^a} \partial_\mu \Phi(x) + \delta\Phi(x) + \mathcal{O}(\epsilon^2). \tag{21}$$

There is some popular convention

$$\delta\Phi(x) = -i\epsilon^a G_a \Phi(x) \tag{22}$$

where G_a is some differential operator realization for the algebra of the symmetry group acting on fields.

Now to get the idea we consider the momentum operator (which generates translations)

$$P : x^\mu \to x^\mu - \epsilon^\mu \tag{23}$$

and we can easily read off that

$$G_\mu(P) = i\partial_\mu \,. \tag{24}$$

Continuing in this fashion for the full conformal group one finds:

$$\begin{aligned} \big[P_\mu, \Phi(x)\big] &= i\partial_\mu \Phi(x)\,, & (25)\\ \big[M_{\mu\nu}, \Phi(x)\big] &= i\big(x_\mu\partial_\nu - x_\nu\partial_\mu\big)\Phi(x)\,, & (26)\\ \big[D, \Phi(x)\big] &= i\big(-\Delta + x^\mu\partial_\mu\big)\Phi(x)\,, & (27)\\ \big[K_\mu, \Phi(x)\big] &= i\big(x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + 2x_\mu\Delta\big)\Phi(x)\,. & (28) \end{aligned}$$

We see that a primary field is also not invariant under special conformal transformations.

3 Correlation Functions of Conformal Fields

The conformal symmetry severely restricts the functional form of correlation functions of primary fields as follows. The one point function vanishes except for the identity operator

$$\langle \mathcal{O}_\Delta(x)\rangle = \delta_{\Delta,0}\,. \tag{29}$$

Indeed, from translation invariance it must be x-independent and from (18) under scale transformations, it must have dimension zero. One concludes that the only non-vanishing one point function is for the identity operator. Alternatively one might say that ay non-zero vev for nono-trivial operator will spontaneously break conformal invariance.

3.1 Two Point Functions

Consider the two point function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle \tag{30}$$

where $\phi_i(x_i)$ is a primary field of dimension Δ_i inserted at position x_i^μ. Now demanding that the transformation (18) holds within quantum correlation functions gives

$$\langle \phi_1'(x_1')\phi_2'(x_2')\rangle = \left|\frac{\partial x_1'}{\partial x_1}\right|^{-\Delta_1/d} \left|\frac{\partial x_2'}{\partial x_2}\right|^{-\Delta_2/d} \langle \phi_1(x_1)\phi_2(x_2)\rangle\,. \tag{31}$$

In other words, this correlator is a (not necessary regular) function of the relativistic invariant

$$x_{12} = |x_1 - x_2| \tag{32}$$

which transforms in a prescribed way under dilatation and special conformal transformations.

Under dilatations we thus demand

$$\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle = \lambda^{-\Delta_1-\Delta_2}\langle \phi_1(x_1)\phi_2(x_2)\rangle \tag{33}$$

from which we get

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,. \tag{34}$$

This is already quite a simplification. Under special conformal transformation we have

$$K_\mu:\ x_{ij} \to \frac{x_{ij}}{(1+2a_\mu x_i^\mu + a^2 x_i^2)^{1/2}(1+2a_\mu x_j^\mu + a^2 x_j^2)^{1/2}} \tag{35}$$

and this in effect block diagonalizes the two-point functions:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{\Delta_1}}\,, \tag{36}$$

where we have also normalized the fields such that the numerical coefficient is one.

Once again, we recall that this does not fix the correlator completely since quite generally, Δ_i must be computed in perturbation theory, even for very supersymmetric theories it is not protected. In many two-dimensional theories with even larger symmetry groups, one can often compute the spectrum of dimensions exactly.

3.2 Three Point Functions

The functional form of the three point functions is also fixed using the same tools as for the two point functions. Poincaré invariance under covariance under dilatations restricts a three point function to the form

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^\alpha x_{23}^\beta x_{31}^\gamma} \tag{37}$$

with

$$\alpha+\beta+\gamma = \sum_{i=1}^{3}\Delta_i\,. \tag{38}$$

The special conformal transformations give

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}\,. \tag{39}$$

The set of constants C_{ijk} are referred to as the *structure constants* and their exact value requires taking into account the field normalization which lead to (36). These structure constant appear in the operator product expansion but in general the full set of structure constants involves fields of arbitrary spin, not just scalar fields.

While this is an impressive simplification one should note that for the allegedly *simplest quantum field theory* namely $\mathcal{N}=4$ SYM in four dimensions the computation of the dimension of single trace primary operators has been has occupied an army of physicists for nearly a decade (see [1] and lectures at this school). The structure constants are just being seriously attacked quite recently.

3.3 Four Point Functions

Much is known about four-point functions in CFT's, the initial application of conformal invariance allows for some undetermined functional dependance in terms of the so-called *conformal cross ratios*

$$u = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad v = \frac{x_{14}x_{23}}{x_{13}x_{24}}. \tag{40}$$

Restricting to four dimensions, there are some very strong results available [2]. We define

$$u^2 = z\bar{z}, \quad v^2 = (1-z)(1-\bar{z}) \tag{41}$$

and using the operator product expansion one can write

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_k C^k_{14} C^k_{34} \mathbf{G}^{12,34}_k(x_1,x_2,x_3,x_4) \tag{42}$$

where

$$\mathbf{G}^{12,34}_k(x_1,x_2,x_3,x_4) = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} \overline{G}^{12,34}_k(u,v) \tag{43}$$

and

$$\begin{aligned} \overline{G}^{12,34}_k(u,v) \;=\; & \left(-\frac{1}{2}\right)^l \frac{z\bar{z}^{(\Delta_k-l)/2}}{z-\bar{z}} \Big[z^{l+1} {}_2F_1\left(\frac{\Delta_k-\Delta_{12}+l}{2}, \frac{\Delta_k+\Delta_{34}+l}{2}, \Delta_k+l, z\right) \times \\ & {}_2F_1\left(\frac{\Delta_k-\Delta_{12}-l-2}{2}, \frac{\Delta_k+\Delta_{34}-l-2}{2}, \Delta_k+l-2, \bar{z}\right) - (z \leftrightarrow \bar{z}) \Big] \end{aligned} \tag{44}$$

One can check using (35) that the factor preceding $\overline{G}^{12,34}_k(u,v)$ in (43) has the correct transformation rule for a four-point function. Then clearly $\overline{G}^{12,34}_k(u,v)$ is conformally invariant.

4 Comments

Probably the three most canonical operators in any given QFT are the identity, the stress tensor and the set of conserved currents arising from global symmetries.

- the identity operator has $\Delta = 0$
- the stress tensor $T_{\mu\nu}$ has dimension d by dimensional analysis
- conserved currents have dimensiosn $d-1$

References

[1] N. Beisert, C. Ahn, L. F. Alday, Z. Bajnok, J. M. Drummond, *et al.*, "Review of AdS/CFT Integrability: An Overview," 1012.3982.

[2] F. Dolan and H. Osborn, "Conformal four point functions and the operator product expansion," *Nucl.Phys.* **B599** (2001) 459–496, hep-th/0011040.